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Partitional Analysis and Rhythmic Partitioning: Mediations between Rhythm and Texture

1. Introduction

Partitional analysis (henceforth *PA*) is an original theory with some new concepts and tools, aiming the application of abstractions derived from the mathematical theory of integer partitions to compositional practices and musical analysis. It has been developed since 2003 (Gentil-Nunes & Carvalho 2003) and has resulted in some published papers, thesis, compositions and analyses in Brazil (for a list of productions, see Gentil-Nunes 2009).

The fundamentals of PA are established in a first step through a mediation between Wallace Berry's textural analysis (1976: 184–199) and the mathematical theory of integer partitions (Andrews 1984; Andrews & Eriksson 2004).

Berry proposed the codification of textural progressions through comparison of distinct profiles of component parts of a musical discourse. The independence or interdependency between concurrent parts constitutes the so called "textural configurations", whose discursive motion delineates the textural progression and recession curves.

Mathematical theory of integer partitions works as an ideal model for developing such an exhaustive taxonomy of that kind of configurations, following criteria that may vary according to the analytical desired focus.

Partitional analysis is constructed by the detailed observation of the application by those very criteria, comparing concurrent parts by pairs. Binary comparison works through simple algorithms, defining collaboration or opposing relations between parts. The process results in two complementary indexes – *agglomeration* (***a***) and *dispersion* (***d***) – delineating a unique profile for each textural configuration. Each pair of indexes can be plotted in a phase space graphic that expresses the topology of the field of choices at disposal for the composer.

The successive decisions delineated by the composer in scores (for instance) form a trajectory. This field is called *partitiogram* and works as an inventory of all possible configurations relative to a given number of elements (for example, instruments, voices, sounds, lines). It also expresses the topology of the relations between the configurations, according to their metric distance in the graphic.

The plotting of both indexes through time generates the *indexogram*, where the interactions between the two resulting lines can be read as four basic progressions (resizing, revarying, transference and concurrence), each one with distinct functions.

Rhythmic partitioning is the computational application directly derived from Berry's work that uses the onset points and durations as inputs for graphic plotting.

Analyses of rhythmic partitioning were made with the software PARSEMAT (Gentil-Nunes 2013/2004), programmed by this author to streamline the task from MIDI files. Some concert music (from Bach, Beethoven, Schoenberg, Webern, Boulez, Ferneyhough, among others) has already been processed.

The results are very expressive and point out to an effective role of rhythmic partitioning as a subjacent organizing principle under the musical discourse, as well as an intimate implication of rhythmic concurrent profiles on texture and form.

Rhythmic partitioning is part of a broader research that observes the application of the abstractions derived from PA to various fields of musical composition (melodic structures, timbre, form) with regard to the vertical interaction between elements. In that sense, PA surpasses far beyond the range of Berry's proposal, thus possibly constituting a general theory of musical verticality. Furthermore, the possibility of a perfect homology between heterogeneous parameters and the mapping between them can also be a exciting possibility for composition and musical analysis.

2. Textural analysis

Musical texture is a theoretical field covered by Wallace Berry in his book *Structural Functions in Music* (1976). In the chapter about that subject (p. 184–199), Berry defines texture as a musical parameter *"… conditioned in part by the number of those components sounding in simultaneity or concurrence, its qualities determined by the interactions, interrelations, and relative projections and substances of component lines or other sounding factors."* (Ibid.: 184).

Berry's conception about texture is dualist. "Density" represents the quantitative aspect of the configurations (based on the number of concurrent sounding components – the "density-number") and the level of compression of components in a given intervallic range ("density-compression"). On the other side, the interactions and interrelations between components will constitute the quality of texture, departing from the variations on independence and interdependence between components.

From that duality, Berry establishes a differentiation between the raw "sounding component", taken alone as quantity, and the "real component", considered as a result of the interactions between sounding components:

> "*Two lines moving in parallel 3rds. may in an important sense be said to constitute a single real textural factor consisting of two components. At any point at which differentiation is established – in rhythm, in direction of motion, in the distance of motion, or in any other sense – a texture initially consisting of a single real factor (of two sounding components) becomes a texture of two real factors (or at least progresses in the direction of such differentiation).*" (Berry 1976: 186)

The movement of the sounding components, their sudden appearance or disappearance, and the coincidences and contrapositions of their articulations will form what Berry call "textural progressions and recessions".

As an example, Berry presents a musical excerpt of Milhaud (1934) where the independence and interdependence relations are represented by piles of numbers, referring to the thickness of each real component (Figure 1).

Figure 1. Milhaud (1934) – *A peine si le coeur vous a considerées, images et figures*, excerpt: real components (Berry 1976: 187–188)

According to Berry, "*In the example, there is progressive development of textural complexity toward m. 4 and recessive decline in that complexity (toward textural accord and simplicity) in approach to the cadence at m. 7*" (Ibid., p. 186). The author also draws attention to the fact that the density-number has its own curve, described by the vector <1 2 3 4 4 4 4> and independent of quality curves delineated by the progressions of configurations (Figure 2).

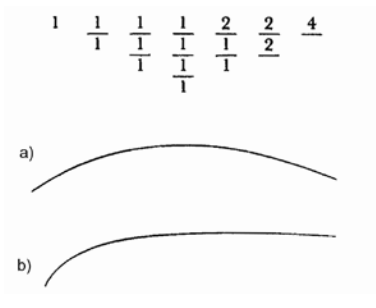

Figure 2. Qualitative textural progression and recession (a) and quantitative progression (b) in Milhaud 1934 (Berry 1976: 188)

Most of the value of Berry's analysis remains in the demonstration of viability of a systematization of textural thought in a more objective way comparing to the current compositional pedagogy. Nevertheless, the analysis itself has some drawbacks that were observed in a former paper from the present author, referring to the "observation window" and some motivic features that were left out by Berry (Gentil-Nunes 2006).

3. Theory of integer partitions

According to Andrews (1984: 149), "*The theory of partitions is an area of additive number theory, a subject concerning the representation of integers as sums of other integers*".

Following definition, number five, for example, has seven partitions – ways by which it can be represented by the sum of other integers (Figure 3).

5
4 + 1
3 + 2
3 + 1 + 1
2 + 2 + 1
2 + 1 + 1 + 1
1 + 1 + 1 + 1 + 1

Figure 3. Partitions of number five

Departing from that operation, Andrews defines the ***p*** function as the denotation of the number of partitions of ***n***. In the given example, p(5) = 7. For each integer there is a distinct number of partitions of it. The main goal of partition theory is quantifying and enumerating of partitions of a given integer, and enunciating the partitional identities. These identities are congruencies established between partitioning, accomplished by different pre-defined conditions. One example given by Euler (1748; the first author that suggests partitioning in that sense) can be expressed in this way: the number of partitions of an integer ***n*** in which all parts are odd is equal to the number of partitions of n which all parts are distinct. Andrews and Eriksson (2004: 3) mark that "*it is an intriguing fact that there are so many different and unexpected partition identities*".

Representation of partitions in the specialized literature has two basic forms:

a) Standard or lexicographic – the parts are grouped in vectors, in full and in lexicographic order (Zohgbi e Stojmenović, 1998: 320–321), For instance, the partitions of five are represented by <11111, 2111, 221, 311, 32, 41, 5>.

b) Representation by multiplicity or abbreviated – "*a more concise notation, where the number of each part is registered in an exponent, so that 7 + 7 + 5 + 1 + 1 + 1 + 1 is written* $7^2 5^1 1^4$" (Andrews e Eriksson 2004: 34). In this case, by convention, the parts are presented in decreasing order, inversely to the standard representation.

Beyond the numeric way, partitions can also be represented by graphic diagrams, which can be, according to Andrews (1984: 6), "*another effective elementary device for studying partitions ...*".

The most important graphic representation of partitions is Ferrers's diagram or Young's diagram (Figure 4). On both, parts are represented by dots or squares, respectively, distributed on the plane according to their size (horizontal dimension) and multiplicity (vertical dimension).

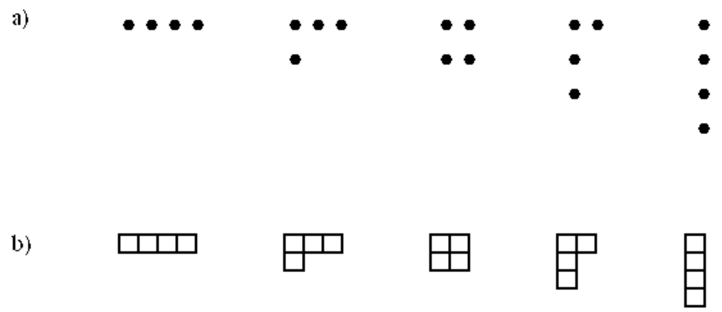

Figure 4. a) Ferrers's diagram and b) Young's diagram for partitions of number four (4, 31, 2^2, 21^2, 1^4)

Young's lattice is the representation of all Young's diagrams ordered by inclusion relations. In this kind of relation, each block precedes and connects to the one in which can be graphically included, with the left superior edge coincident (Figure 5).

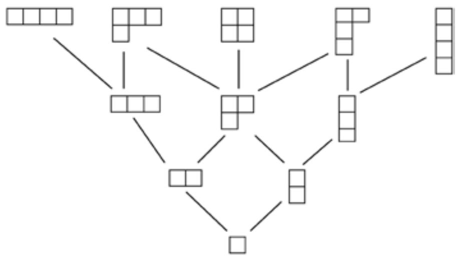

Figure 5. Young's lattice limited to partitions for n ≤ 4 (Andrews e Eriksson 2004: 108)

In the development of PA, a structure similar to Young's lattice emerged as relations between partitions were discovered and registered.

4. Binary relations

The mediation between textural analysis and theory of partitions is constructed departing from a basic and simple concept, called "binary relations". These relations play an important role on the very definition of the partitions, seeing from a musical point of view.

The concept can be better understood taking a glance on some traditional techniques of basic textural training, like exercises of Harmony and Counterpoint. Every time the teacher asks the students to find matches between parts – searching, for instance, for parallels fifths or octaves, they have to check the parts exhaustively by pairs. For a mixed choir (SATB), there are six pairs to check out – BT, BA, BS, TA, TS, AS (Figure 6). In other words, combinations of four – two by two.

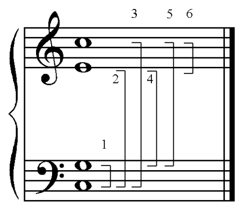

Figure 6. Binary relations, four parts (Gentil-Nunes 2009: 33)

The pertinent observation to be made is that Berry's textural configurations are defined by the same operations – the matching between parts, following some criteria. The filter – rhythmic congruence between points of attacks and durations – is the key to define what will be grouped or not. But the total number of binary relations remains the same for every density-number.

In successive textural configurations, the components will, at each moment, actualize their relations. In the same way that textural configurations form quantitative and qualitative curves, binary relations are renewed, creating an autonomous movement. The numbers of congruent and not congruent binary relations are accounted as two indexes, respectively called agglomeration (***a***) and dispersion (***d***) indexes. For each textural configuration, a pair (***a, d***) is assigned.

The progressive movement of binary relations and the pairs (***a, d***) in time is observed in a very simple example – an excerpt from Mozart (1877), where textural configurations are presented in an elementary way (Figure 7).

Figure 7. Mozart, *Eine Kleine Nachtmusik*, K. 5, excerpt: partitions (Gentil-Nunes & Carvalho 2003: 43)

The observation of the string quartet, with the focus on the congruence between points of attack, can show that, at the moment of articulation of each textural configuration, an exclusive disposition of binary relations is established (Figure 8, where the solid lines represent agglomeration and the dotted ones represent dispersion). In other words, to the partitions 2, 2^2, 13 e 4 correspond, respectively, the pairs ***(a, d)*** – (1,0), (2,4), (3,3), (6,0).

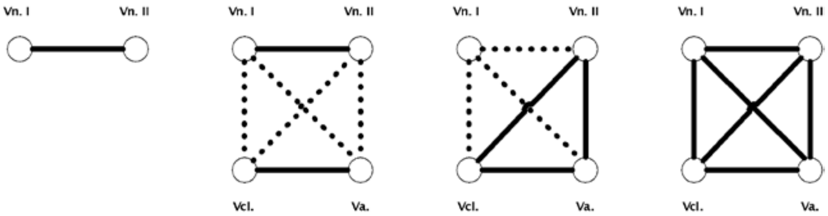

Figure 8. Binary relations in 2, 2^2, 13 e 4 (Gentil-Nunes, 2009: 36)

Berry, in his essay about texture, doesn't even propose the listing of all possibilities of partitioning of a given density-number. In fact, there's not yet, inside the music theory field, an exhaustive taxonomy of all the possibilities of partitioning of a given density-number, similar to, for instance, the taxonomy made by Allen Forte (1973), related to pitch classes. This is a necessary condition for contextualization of each fractioning (partitioning) inside a significant global system.

For that reason, there's not yet, too, a systematization or study of techniques of conscientious handling of these elements. We can assume here that all this relations and progressions are part of a limited repertoire constituted by compositional gestures, articulated automatically, and probably repetitiously, by generations of composers. This situation excludes the use of some others (probably new) combinatory possibilities, restricting also the knowledge about the ones that are already in use.

The agglomeration and dispersion indexes are the key to the construction of a graph that can represent that taxonomy – the *partitiogram*.

5. Partitiogram

Once the partitions are finite and known as mathematical entities, and once it's possible to attribute to each of it a pair of indexes that reflects its grade of internal agglomeration and dispersion, it is convenient to plot the partitions in a plane. The *partitiogram* works as a topology of the partitioning field (Figure 9, where n ≤ 9), an exhaustive taxonomy of the possibilities of ***n*** and constitute, too, a phase space, in the sense of "*set of elements conditioned by independent variables that evolve in time*" (Bergé, 1994: 91).

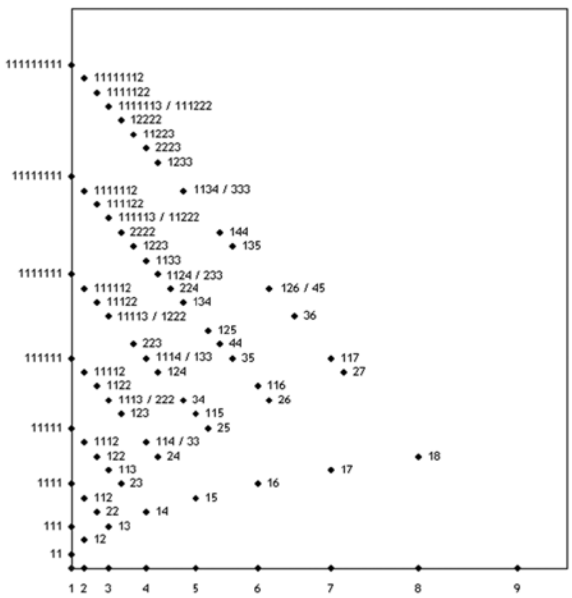

Figure 9. Partitiogram for *n* ≤ 9 (Gentil-Nunes and Carvalho 2003: 48). Graph generated by PARSEMAT (Gentil-Nunes 2004)

The partitiogram is also a representation of the lexical-set[1] of a given number. Namely, it presents the repertoire of all possible textural configurations for a density-number. In that sense, the concept of lexical-set is convenient for musical application, once the partitioning, in this case, is devised for compositional purposes. It represents too the possibilities of the medium (ensemble, instrument, computational resources).

We can think about the partitiogram as a kind of Young's lattice, positioned obliquely, with its right diagonal side parallel to the x-axis. However, some important differences are noted. Here, the partitions have a precise geographical organization. The distances between them are significant and quantized, which did not occur in Young's lattice. We can measure the difference about relational content between two partitions by the intervenient metric interval. For instance, there's a closer relation between the partitions [2 7] and [1 1 7] than the partitions [3 6] and [1 1 7], although there's a simple and symmetric neighborhood between the three in the Young's lattice.

The partitiogram inherits from the function *p(n)* its fractal irregularity and isn't consistent graphically with exponential progressions, although it has some kind of predictability. Furthermore, the distribution of partitions is very unbalanced, with a remarkable predominance of more dispersed partitions near the y-axis.

In a broader view, it is observed that the x-axis increases toward massive, choral textures, while the y-axis increases toward more linear and polyphonic textures.

6. Partial orders inside partitiogram

As the partitiogram resembles the Young's lattice, which is a partially ordered set, it's possible to get a reading from partial orders embedded in the structure of partitiogram and set conjunctions and disjunctions in accordance with these orders. Another way to extract partial orders is through comparison between indexes themselves, using the internal organization of the partitions, represented by pairs (***a, d***) to find conjunctions and disjunctions.

In this paper, four partial orders between partitions and between indexes are presented. The partitional orders (items a, b, c, d) involve real elements (actors), while the order of reglomeration (item e) is established from the internal structure of the partitions. For each item is assigned a letter for further indication of the systems in abbreviated form.

Resizing (m) derives from the relationship of inclusion, which is the usual order of the Young's lattice. However, only refers to transactions where there is a change in the horizontal, or both horizontal and vertical dimensions. In terms of texture, this movement corresponds to a unilateral actor behavior: a single element becomes more or less dense.

Revariance (v) derives from the relationship of inclusion, such as the resizing operation, but only refers to transactions where there is change in the vertical dimension. This is an unilateral act, where a new element emerges or an existing unitary element disappears.

Transfer (t) defined as a combined and complementary modification of the horizontal and vertical dimensions. This means that there is a collaborative relationship between actors in order to maintain the constancy of the density-number. When a part becomes thin, others appear to compensate for the loss of density and vice versa. This is the relationship that prevails in traditional partitional discourse.

Concurrence (c) consists of a parallel movement of both dimensions – a simultaneous movement of the indexes ***a*** and ***d,*** in the same direction. The addiction of a new unitary part is accompanied by the densification of another, and vice versa. The concurrence causes higher contrasts and is prevalent in Darmstadt style.

Regglomeration (r) is defined when the dispersion index between partitions is fixed and only the index of agglomeration is articulated.

7. Indexogram

The *indexogram* is another way of representing the evolution of the agglomeration and dispersion indexes, plotted against a temporal axis. Once both indexes are positive, they were arranged in a mirrored representation, where the agglomeration is plotted negatively. Thus, the distance between the points defined by the contents also becomes a visual measure of the density-number (Figure 10).

[1] Lexical set is a concept of Partitional Analysis. Lexset (***n***) is the union of the sets formed by all the integer partitions of 1 to ***n***.

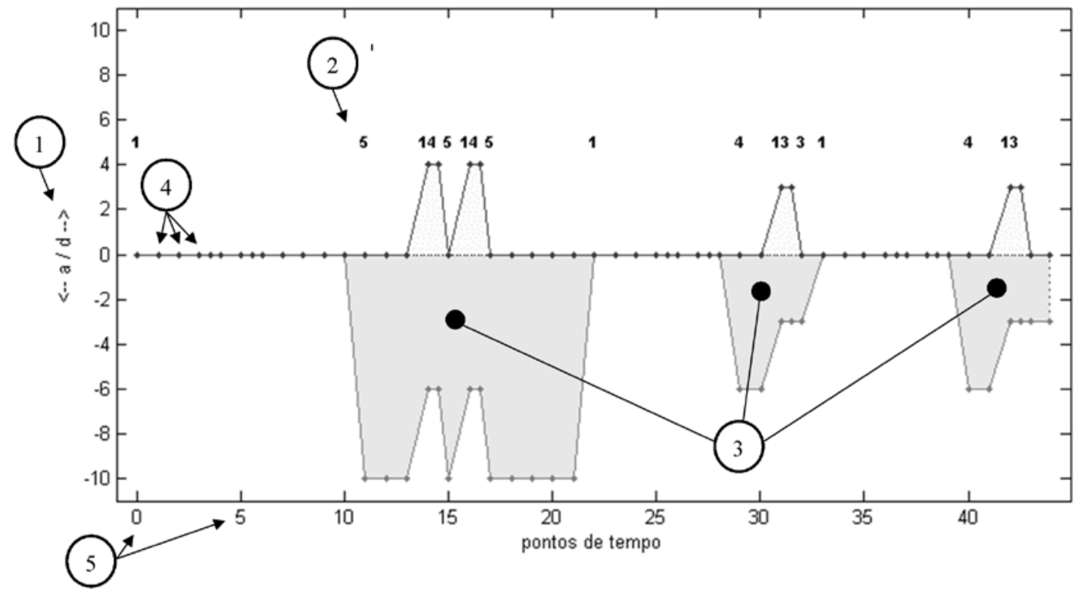

Figure 10. Indexogram elements: 1) abbreviated labels for agglomeration and dispersion indexes; 2) representation of the partitions; 3) "bubbles"; 4) indication of attack points; 5) time points (***beats***) (Gentil-Nunes 2009: 53). Graph generated by PARSEMAT (Gentil-Nunes 2004)

The purpose of indexogram is quite different from partitiogram. The indexogram highlights the movements of the indices over time and thus has a homology with the score, allowing the comparison with the musical text more directly. It brings new information about the partitions, which the partitiogram does not show at all – for instance, their durations.

The linear movements between the indexes (parallel, direct, contrary and oblique, similar to melodic counterpoint movements) have direct correspondence with the paths traced by the composer in partitiogram. The interaction between the indexogram and the partitiogram can also be used to read the behavior of partitions in time, enriching, mutually, the meaning of both tools, thus constituting an integrated system.

8. Conclusions

Partitional Analysis is under construction. For now, it could develop and explain partially the close relationship between concurrent rhythmic profiles and texture. At the moment, the research unfolds through various branches within the MusMat Research Group[2].

One of the directions is the application of PA to linear structure, aiming to objectify the melodic texture. The concept of line is derived from schenkerian theory and allows the homology with the rhythmic partitioning.

Other project developed at Federal University of Rio de Janeiro is dedicated to applying partitional analysis to orchestration of Debussy's works. Comparisons between rhythmic partitioning and orchestral indexograms allowed the visualization of essential characteristics of the orchestration of Debussy, such as the enclosure of textural elements within each orchestral group.

References

Andrews, George. *The theory of partitions*. Cambridge: Cambridge University, 1984.

Andrews, George e Eriksson, Kimmo. *Integer partitions*. Cambridge: Cambridge University, 2004.

Bergé, Paul. *Dos ritmos ao caos [From the rhythms to the chãos]*. Translation by Roberto Leal Ferreira. São Paulo: UNESP, 1994.

Berry, Wallace. *Structural functions in music*. New York: Dover, 1976.

Euler, Leonhard. *Introduction to Analysis of the Infinite*. New York: Springer-Verlag, 1748.

Forte, Allen. *The structure of atonal music*. New Haven: Yale University, 1973.

Gentil-Nunes, Pauxy e Carvalho, Alexandre. Densidade e linearidade na configuração de texturas musicais [Density and linearity in configuration of musical textures]. *Anais do IV Colóquio de Pesquisa do Programa de Pós-Graduação da Escola de Música da UFRJ [Annals of IV Research Colloquium of Post-Graduation Programo of Music School of Federal University of Rio de Janeiro]*. Rio de Janeiro: UFRJ, 2003.

[2] Musmat Research Group is based at the Federal University of Rio de Janeiro and focus on the research of mathematical modeling applied to composition and musical analysis (www.musmat.org).

Gentil-Nunes, Pauxy. *PARSEMAT – Parseme Toolbox Software Package*. Rio de Janeiro: Pauxy Gentil-Nunes. 2004/2013. Acessible in www.musmat.org/downloads

Gentil-Nunes, Pauxy. *Funções sociais dos números e composição de música de Concerto [Social functions of numbers and the composition of concert music]*. Rio de Janeiro: UNIRIO, 2006a.

Gentil-Nunes, Pauxy. *Análise particional: uma mediação entre composição musical e a teoria das partições [Partitional analysis: a mediation between musical composition and the theory of partitions]*. Tese (Doutorado em Música). Rio de Janeiro: Programa de Pós-Graduação em Música, Centro de Letras e Artes, Universidade Federal do Estado do Rio de Janeiro, 2009.

Milhaud, Darius. A peine si le coeur vous a considerée images et figures. *Six sonnets*. Paris: Alphonse Leduc, 1934.

Mozart, Wolfgang Amadeus. *Eine kleine nachtmusik*. String quartet (2 violins, viola, violoncello). Leipzig: Breitkopf & Härtel, 1877.

Zohgbi & Stojmenovic. Fast algorithms for generating integer partitions. *International Journal of Computer Mathematics*, 70, pp. 319–322, 1998.

Santrauka

Partityvinė analizė ir ritmo dalijimas: ritmo ir faktūros sąveika

Partityvinė analizė – tai originali teorija su naujomis koncepcijomis ir įrankiais, skirtais abstrakcijų, kilusių iš matematinių sveikojo skaičiaus dalybos teorijos, taikymui kompozicinėje praktikoje ir muzikos analizėje. Ji plėtojama nuo 2003 m., ir šia tema Brazilijoje jau parengta mokslo tiriamųjų straipsnių, disertacijų, kompozicijų. Ji grindžiama Wallace'o Berry faktūrine analize (1976) ir matematine sveikųjų skaičių dalybos teorija. Berry siūlė koduoti faktūrines progresijas lyginant individualius muzikinio diskurso komponentų dalių profilius. Savarankiškumo ar tarpusavio priklausomybės ryšiai tarp dalių sudaro vadinamąsias „faktūrines konfigūracijas", kurių diskursyvus judėjimas apibrėžia faktūrinės progresijos ir recesijos kreives. Matematinė sveikųjų skaičių dalybos teorija – tai puikus modelis plėtojant išsamią tokių konfigūracijų taksonomiją pagal kriterijus, kurie gali įvairuoti priklausomai nuo pasirinktos analizės krypties. Partityvinė analizė konstruojama atidžiai stebint tokių kriterijų taikymą, lyginant sutampančias dalis poromis. Binarinis lyginimas veikia pasitelkus paprastus algoritmus, nustatančius bendrumus arba priešingumo ryšius tarp dalių. Šio proceso rezultatas – du vienas kitą papildantys rodikliai: aglomeracija (*a*) ir dispersija (*d*), kurie atspindi unikalų kiekvienos faktūrinės konfigūracijos profilį. Kiekviena rodiklių pora nubraižoma fazės grafike, kuris išreiškia pasirinkimų lauko topologiją kaip kompozitoriaus pasirinkimą. Nuoseklūs kompozitoriaus partitūrose brėžiami sprendimai suformuoja tam tikrą trajektoriją. Šis laukas vadinamas partitiograma – tai visų galimų konfigūracijų visuma, susijusi su nustatytu elementų (pvz., instrumentų, balsų, garsų, linijų) skaičiumi. Jis taip pat išreiškia santykių tarp konfigūracijų topologiją, atsižvelgiant į jų metrinį atstumą grafike. Abiejų indeksų nubrėžimas laike sukuria indeksogramą, kurioje santykis tarp dviejų linijų gali būti įvardijamas keturiomis bazinėmis progresijomis (dydžio keitimas, perkeitimas, perkėlimas ir sutapimas), o kiekviena iš jų turi savą funkciją. Ritmo dalijimas – tai kompiuterinė aplikacija, kilusi iš Berry darbų; ji naudoja pradžios taškus ir trukmes kaip įvesties duomenis grafikams braižyti. Ritmo dalijimo analizė buvo atlikta pasitelkus kompiuterinę programinę įrangą PARSEMAT, sukurtą šio darbo autoriaus siekiant racionalizuoti MIDI failų analizę. Šiuo metu jau yra išanalizuota dalis koncertinės muzikos (Bachas, Beethovenas, Schönbergas, Webernas, Boulezas, Ferneyhough ir kt.). Rezultatai yra labai išraiškingi ir atskleidžia, kad ritmo dalijimas kaip muzikinio diskurso organizavimo principas yra labai efektyvus ir parodo stiprią ritminių profilių įtaką kūrinio faktūrai ir formai.