Priedas

Supplement

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"Music–Mathematics" Interconnections: An Approach through Science (Works by Mathematicians), and an Approach through the Arts (*Musica mathematica* by Rima Povilionienė)

Muzikos ir matematikos tarpusavio sąveika: mokslinis traktavimas (matematikų darbai) ir meninis traktavimas (Rimos Povilionienės „Musica mathematica")

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Abstract

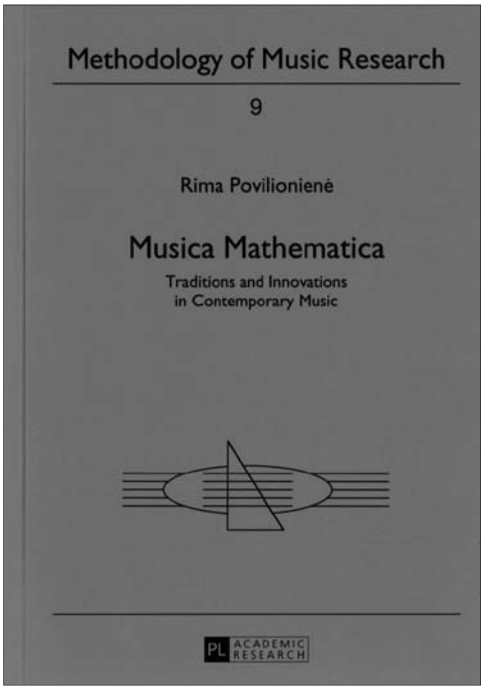
The interactions between music and mathematics stretch back to the classical Antiquity, and beyond. These relations can be classified into several groups: *intrinsic* relations (those coming from mathematical physics of musical instruments, Fourier analysis needed to describe the nature of a single sound, resonance phenomena, damped vibrations); *cultural*, or related to local cultural tradition of harmony (mathematics of tuning, conundrum of consonance-dissonance); *sound technologies* (digital music, synthesis); *technologies of composition* (form, symmetry, constructive tools: semantic-symbolic, formal-constructive); and so on. Yet, the majority of studies about the relation itself were written by mathematicians, who, as a rule, adhered to an *evolutionary* point of view. The latter attitude is apt in natural and physical sciences. It is not a universal paradigm though, especially dealing with conscience, spirituality, Arts, society. Music develops in a non-evolutionary way, at least its essence, not the technology itself (craftsmanship of instruments, advancement of musical theory, diversification of social stimuli, and so on). Additionally, literature on the last item of the classification (technologies of composition) was sparse. The study *Musica Mathematica: Traditions and Innovations in Contemporary Music* (Peter Lang Academic Research, 2016) by Rima Povilionienė presents the basics of the theory from an *artistic* standpoint. Moreover, it is written in a way to provide a useful tool for composers and musical theorists. Therefore, the aim of this paper is twofold. Firstly, aiming at those musicologists who have no mathematical training at university level, I describe several cross-sections of music where mathematics does intrinsically manifest, putting a particular emphasis on subjects relevant to composers. The second aim is to review the aforementioned monograph. The explanation of the role of *Musica Mathematica* in the context of the whole related literature might be most useful to composers and those wishing to seek connections between music and mathematics, having in mind fundamental similarities of these two subjects, but also being aware of their fundamental differences.

Keywords: scales, temperament, tuning, commas, wave equation, instruments, digital music, synthesis, combinatorics, algorithmic composing, numerology, constructive composing, semantics.

Anotacija

Muzikos ir matematikos tarpusavio ryšiai siekia antiką ir dar giliau. Šie ryšiai gali būti klasifikuojami į keletą grupių: *savybingi* (kylantys iš matematinės fizikos, kuri aprašo muzikos instrumentus, iš Furjė analizės, aprašančios vieno garso prigimtį, iš rezonanso reiškinio, slopinamos vibracijos); *kultūriniai*, arba susiję su lokaliąja harmonijos tradicija (derinimų matematika, konsonanso–disonanso problematika); *garso technologijos* (skaitmeninė muzika, sintezė); *komponavimo technologijos* (forma, simetrija, konstruktyvusis komponavimas: semantinis simbolinis, formalusis konstruktyvusis) ir taip toliau. Visgi didžioji dalis monografijų, skirtų šiai sąveikai, buvo parašyta pačių matematikų, kurie dažniausiai laikėsi *evoliucinio* požiūrio taško. Tokia prieiga yra tinkama gamtos ir fizikiniuose moksluose. Bet tai nėra universali paradigma, ypač jei tyrinėjama sąmonė, dvasingumas, menai, visuomenė. Muzika vystosi neevoliuciniu būdu, jei kalbėtume apie pačią esmę, o ne technologijas (instrumentų gamybos meistrystė, muzikos teorijos vystymasis, visuomenės sukeliamų stimulų diversifikacija ir t. t.). Rimos Povilionienės monografija „Musica Mathematica: Traditions and Innovations in Contemporary Music" (Peter Lang Academic Research, 2016) atskleidžia teorijos pagrindus būtent iš šio *meninio* požiūrio taško. Be to, monografija parašyta taip, kad būtų naudingas įrankis kompozitoriams ir muzikos teoretikams. Taigi, šio straipsnio tikslas yra dvejopas. Pirma, orientuojantis į muzikologus, kurie neturi universitetinio lygio matematinio pasiruošimo, aprašyti keletą muzikos pjūvių, kur matematika esmingai pasireiškia, ypač akcentuojant vietas, naudingas kompozitoriams. Antrasis tikslas yra apžvelgti minėtą monografiją. Apibūdinti vaidmenį, kurį „Musica Mathematica" vaidina susijusios literatūros kontekste, turėtų būti naudingiausia kompozitoriams ir tiems, kurie ieško pamatinių ryšių tarp muzikos ir matematikos, turint galvoje esminius panašumus, bet gerai suvokiant ir esminius skirtumus.

Reikšminiai žodžiai: gamos, temperacija, derinimas, komos, bangos lygtis, instrumentai, skaitmeninė muzika, sintezė, kombinatorika, algoritminis muzikos kūrimas, numerologija, konstruktyvus muzikos kūrimas, semantika.



Rima Povilionienė, *Musica Mathematica: Traditions and Innovations in Contemporary Music*, [Series Methodology of Music Research,] Vol. 9, ed. Nico Schuler. Frankfurt am Main, New York: PL Academic Research (Peter Lang), 2016, 288 p. ISBN 978-3-631-71381-5, ISSN 1618-842X.

1. Music and mathematics: two attitudes

The issue of identifying the time period (from the anthropologic point of view) when interactions between scientific disciplines and Arts first occurred is meaningless. Even if we limit ourselves to the physical Science of Mathematics and the Art of Music, the very first interactions cannot be dated. Truly, this symbiosis is said to be the oldest symbiosis between two fields of knowledge and creativity. The connection stretches beyond classical Antiquity, up to civilizations in Mesopotamia and Egypt. If compared to other physical and natural sciences, like physics, biology, chemistry, astronomy, Earth sciences, mathematics in its scope, the task and aesthetic view is much closer (and sometimes can be classified, at least informally) as one of the Humanities, having special proximity to philosophy, religion, linguistics, visual and performing arts. On the other hand, music is the branch of Art with the deepest and most intrinsic mathematical roots.

It is therefore not surprising that an impressive amount of books and monographs appeared with both names "Music" and "Mathematics" in their title. In recent decades, we have witnessed a substantial increase in the interest of these connections. It is out of the scope of this paper to describe all items in the bibliography list individually. Each author(s) have their own personal view, inclinations, insights, musical and mathematical background, and artistic training. Myself, though trained as a professional mathematician who specializes in pure mathematics, I have devoted a substantial amount of time to study music, poetry, literature, history, language, religion, and mythology. Therefore, there are certain issues related to each of the aforementioned author's worldview. This is the right point to indulge the reader in a brief explanation of the precise goals of this survey.

Surely, many cross-sections unite music and mathematics. The majority of these are very well known, so I will mention only a few current topics. Namely, those where musicians and people unfamiliar with routines in physical sciences tend to oversimplify or, most commonly, even misjudge.

The very roles of *creativity, imagination, intuition, erudition, memory, scholarship, discipline, inspiration,* are not that distinct in Arts and in Sciences. Of course, there are certain issues concerning *individuality, emotionality, psychology, influence, and outreach zones*. For example, this happens due to linguistic, religious, cultural, or geographical barriers. A standard mistake and deep misapprehension of the nature of mathematics is to claim that mathematics amounts to finding algorithms, and the derivation of theorems from a set of given axioms. Quite the contrary is true! As a groundbreaking and pivotal contribution to mathematics and philosophy, Kurt Gödel (1906–1978) proved his immensely consequential Incompleteness Theorem.[1] One of the corollaries of this result is that mathematics needs *insights*, or *intuition,* or an understanding of the Platonic reality of the mathematical world, to advance further. Mechanical applications of axioms amount to only a small fraction of mathematical knowledge. Another step forward was made by Alan Turing (1912–1954), who, building on the work of Gödel, proved his celebrated Halting Problem. This explains, in essence, that algorithmic mathematics is quite limited in its scope due to inner reasons. Roger Penrose (b. 1931), who investigated problems of the human mind as opposed to the mind of computers, wrote an excellent account on Turing's machines.[2] Thus, it is not the creative process itself that distinguishes music from mathematics.

There is, of course, the issue of *musical mistakes* (harmony, voice leading, instrumentation, false authorship – the so called pseudepigrapha, tuning, etc., the majority of which are subject to individual perception and standards of a particular musical epoch) vs. *mathematical mistake*s (logical, therefore objective) which play essentially different roles in both subjects. Yet, to my mind, it is a dichotomy, or principal differences in historic development of these two subjects which severely divorces them.

The development of mathematics is *evolutionary* (in fact, evolution is blind, and some mutations can impair certain abilities; it is better to use the term *progress* here. In this paper, I mean "progress" when saying "evolution").

The development of music is *mythological*, or *creationistic*. Though this terminology may not seem very accurate (I chose the terms rather arbitrarily), this distinction is very well familiar to those working in the Sciences as well as the Arts. Let us draw a few examples. From the point of view of mathematician Leopold Kronecker (1823–1891), the scientific works of Pierre de Fermat (1607–1665) are just embryonic forms of number theory, judging from its state in the 19th century. If a comparison with the evolution of biological species is debatable for many reasons, then a comparison with an evolution of computer programs or technology is apt. Likewise, the geometry of Tullio Levi-Civita (1873–1941) is far more sophisticated than the geometry of Euclid (300 B.C.), and encompasses the latter as a germ.

Meanwhile one cannot indulge in making any comparisons between the religious music of Guillaume de Machaut (c. 1300–1377) and Pavel G. Chesnokov (1877–1944), or the instrumental music of John Dowland (1563–1626) and Béla Bartók (1881–1945). Even the emotional effect of the first known examples of notated music (the clay tablets dated c. 1400 B.C., excavated in the city of Ugarit, written in Hurrian, with musical instructions written in Akkadian)[3] cannot be compared to evocations caused by, say, minimalistic music. Analysis of technological sophistication, as far as compositional, theoretical, and physical technologies are concerned, is legitimate. Nevertheless, "evolution" is a meaningless concept if the issues of irrationality, worldview, and *psyche* are included.[4] In the same vein, alchemy can be viewed as a pseudo-science, which had its own successes (in discovering new chemical elements, like phosphorus, for example), but which was flawed in its essentials, and which later evolved into the sound science of chemistry. But this attitude is greatly misleading. It leaves aside all issues of religion, magic, spirituality, poetry, imagination, mysticism, psychology, and many others, aside.[5]

The incarnation of this phenomenon is the main deficit in all books on music and mathematics, written by professional mathematicians. This cannot be called a drawback, though. The topics discussed in monographs are fundamental. Yet, during the course of many years, it was mathematicians who showed the greatest interest in music, not the other way round.[6] Many, if not all of the authors, had professional training in musical theory, even in a performing art. It is meaningful though, to pose the following question: what is the *target audience* of these books? Should composers be interested in the theory, is it useful as a tool for composition? In some cases, a few – yes, for example, as the music of Michael Harrison shows (see the next Section). In many other cases – no. Moreover, it is undeniable that it is the evolutionary attitude towards the subject which overwhelmingly dominates. It is enough to go through the articles of the *Journal of Mathematics and Music* (Taylor & Francis), or the activities of the Society for Mathematics and Computation in Music (founded in 2006), to be convinced of this *status quo*.

The time was ripe for a scholar, trained foremostly in Humanities and Arts, to make a necessary contribution to the subject of "Music and Mathematics". In the terminology of this section, a scholar which offers a study from a *creationistic* perspective. This implies an endeavor to analyze the same subject, but from an *a priori* perspective, infused with an attitude which is compatible with the philosophy, aesthetics, and history of development of music.

As a first approach, this does not require sophisticated mathematical machinery. Arithmetic, geometry, basic number theory, combinatorics, rudimentary aspects of dynamical systems, cryptography, algorithmic analysis, are more than sufficient. Such approach should incorporate also philosophy, linguistics, religion, architecture, poetry, history, alchemy, and semiotics. A common reoccurring phenomena to explain the existence of religion is to present it as an artifact of the complicated human conscience. Assuming the latter is getting more and more elaborate with time. However, such an outlook is also a reincarnation of an evolutionary approach. Therefore, topics of esoteric, occult, or mystic origin should also not be aliened to investigations of the interconnections of music and mathematics.

The monograph *Musica Mathematica. Traditions and Innovations in Contemporary Music* by Rima Povilionienė (Peter Lang Academic Research, 2016) is an important step in this direction. The author is equipped with a *scientific* worldview, which is compatible with development in the Arts. It is written by a professional musician and musicologist. Minding the dichotomy "scientific evolution – artistic creationism", this monograph is crucial in giving the right basis for musical theorists and musicologists to get acquainted with the concept of *musica mathematica*. I would like to stress that monographs written by mathematicians are not nullified, quite the contrary. Their role is significant and, hopefully, will increase with time, attracting attention from composers and musical theorists. But one should be clear which attitude is being followed, and what are the consequences of this knowledge to compositional processes and esthetics.

Summarizing, the goal of this paper is twofold. Sections 2–6 are written from a view point of mathematical theory of music (in a simplified from), as it stands now in the 21st century. Since this text is intended for musicologists, some of the mathematical formulas can prove to be unintelligible or obscure for those without mathematical training at a university level. This especially concerns formulas which involve calculus or derivatives. Yet, one can ignore the latter completely without any harm. The rest of the formulas and illustrations in this paper, which involve only arithmetic and geometry, should be clear enough to grasp the variety of ways in which mathematics and music are connected.

The second goal of this paper (in Section 7), is to describe the content of the monograph *Musica Mathematica* in greater detail.

2. Scales and temperaments

The current section deals with Western music. However, the music of India, Southeast Asia (Balinese and Javanese Gamelan, Thailand), and so on, all have important mathematical cross-sections, but are topics too wide to be included in this review.

The hegemony of an octave-based (frequency ratio 2:1) temperament with exactly 12 notes in a scale is a consequence of several mathematical facts, and *a posteriori* few coincidences, which, minding that a reasonable scale should have a number of notes in the range, say, from 8 to 30, is not statistically that implausible, as might be expected, but still an event whose occurrence has a relatively small probability. 8–30 notes are needed in order to have scales, modes, functions and a functional hierarchy which have more or less moderate complexity, so that these complications and interrelations are compatible with human short-term memory capacity. All kinds of memory – iconic, short-term, long-term, semantic, and procedural – have effects on comprehension of music, language and sounds, but short-term memory is extremely important to make melodic, harmonic and form connections.[7]

We start from the assumption that we wish our musical scale to be based on an octave, which is a frequency ratio 2:1. There are experimental tunings based on a *tritave* 3:1; for example, Bohlen-Pierce (BP)-Pythagorean, BP-just, and BP-equal temperament scales, but we will leave these aside (Benson 2006: Section 6.7).

The notion of consonance-dissonance is a very old subject. One can mention contributions by Johannes Kepler, Simon Stevin, Roger Bacon, René Descartes, Pierre Gassendi, Jean-Philippe Rameau, Galileo Galilei, Marin Mersenne, Leonhard Euler, Giuseppe Tartini, Jean le Rond d'Alembert, Georg Andreas Sorge (*Music and Mathematics* 2003: 84; Benson 2006: Chapter 4).

However, it was Hermann von Helmholtz who gave the first scientifically sound explanation of the consonance-dissonance phenomenon, with qualifications that all substantial scientific theories should fulfill (Helmholtz 1954): to explain the facts already known to science, and also to theoretically predict new phenomena, which should be confirmed experimentally afterwards. For example, in a celebrated prediction, Helmholtz asserted that a major third (M3) D–F♯ played by a clarinet and oboe sounds much better when the clarinet takes the lower note, while the fourth (perfect fourth, P4) or a minor third (m3) will sound better when the oboe takes the lower note (*Music and Mathematics* 2003: 85). Note that odd harmonics dominate in the spectrum of clarinet (Benson 2006; Section 3.5). *A posteriori*, such theoretical predictions for two spectrally different instruments should not come as a surprise.

The theory of Helmholtz is still being refined, with such crucial ingredients as understanding a critical bandwidth, physiological and neurological mechanisms of a basilar membrane in a cochlea, and also a central nerve system. But it is generally agreed that the phenomenon of consonance-dissonance encompasses both *timbral* (composition of spectra of two or more instruments) and *arithmetic* (arithmetic-algebraic relations among various fundamental frequencies) qualities. These investigations have the following strong consequence (to be clear, we deal with two violins, but the conclusion remains intact for all instruments with harmonic spectrum; that is, for aerophones and chordophones):

> Potentially acceptable tuning should necessarily encompass a perfect octave, and an extremely good fifth.

Of course, one cannot achieve all perfect fifths due to the *comma of Pythagoras*:

$$\frac{3^{12}}{2^{19}} = 1.01364 \ldots \neq 1,$$

the difference between the presented fraction and 1 being approx. 23 cents.

A question whether major thirds or minor thirds are consonant enough in a tuning, turns out to be of a secondary significance: even if one can achieve that major thirds or minor thirds are just in the majority of tonalities (frequency rations 5:4 and 6:5, respectively; for example, there are 8 just major thirds in the standard ¼-comma meantone temperament), these just major thirds are already dissonant to some degree, and their mistuning does not sharply increase a dissonance. Based on a few premises, one can draw a dissonance curve for any pair of instruments. The first (objective) ingredient which is needed is a spectral composition of each of the instruments. Of course, whether one considers resonant frequencies, or the initial transient part, is a completely different matter. The latter is notoriously difficult do describe mathematically (Fletcher, Rossing 2008). The second (to some degree, subjective) ingredient in drawing a dissonance curve is the *sharpness* curve for two sinusoidal sounds. The initial assumption of Helmholtz (he postulated that the sharpness is maximal at around 30 beats per second, independently of the frequency range) is debatable, and was refined many times afterwards with the help of psycho-acoustic studies. But the main premises remain valid. The phenomenon of *beats* is a consequence of a simple, nonetheless, remarkable addition formula for trigonometric functions:

$$\sin \upsilon + \sin \nu = 2\sin\frac{\upsilon + \nu}{2} \cdot \cos\frac{\upsilon - \nu}{2}.$$

This formula has the utmost importance in the theory of harmony, as well as in deriving equations for an oscillating spring or a pendulum. Summarizing, the most important consequence of the dissonance curve technique, and thus the prime goal of any octave-based tuning, is to get fifths (a frequency ratio 3:2) as good as possible.

Now, assume we divide an octave into n equal parts. Suppose, m steps is the closest we can get to the fifth. In mathematical terms, we need to find a rational number $\frac{m}{n}$, such that:

$$2^{\frac{m}{n}} \approx \frac{3}{2},$$

as accurately as possible. In other words, we need to find a rational number, such that:

$$\log_2 \frac{3}{2} \approx \frac{m}{n}.$$

This is a standard problem in Diophantine approximation (Benson 2006: Section 6.2), and it is solved using continued fraction expansion of an irrational number $\log_2\frac{3}{2}$. This continued fraction starts as:

$[0; 1, 1, 2, 2, 3, 1, 5, 2, 23, 2, 2, 1, \ldots]$.

The corresponding sequence of convergents starts from:

$1, \frac{1}{2}, \frac{3}{5}, \frac{7}{\mathbf{12}}, \frac{24}{41}, \frac{31}{53}, \frac{179}{306}, \frac{389}{665}, \frac{9126}{15601}, \ldots$

The only fraction which gives more than 8 and less than 30 notes (as mentioned, to have more notes is of no use due to memory issues, complications of the theory, and JND-related (just noticeable difference) limitations of the auditory system), is 12 notes in the octave, 7 of which give a fifth. It is an extremely good approximation to a just fifth, giving a deviation of only 1.95 cents. This is the first reason for the appearance of a number "12". It is a purely a mathematical fact, with no structure behind it – just a particular term in the series of denominators associated with a particular irrational number.

Surprisingly, the number 12 appears from a completely different standpoint, and this coincidence seals the hegemony of a 12-part scale, leaving all other theories in the periphery. There is yet another additional *a posteriori* auspicious consequence which makes 12 even more dominant. The second source for 12 notes arises from functional considerations. In contrast to the previous case, where we cared only about the second (a just octave) and the third (a just fifth) overtones, now we will accommodate a fifth overtone (a major third) into the theory. And so, the setup starts from the tonic (1:1), the dominant (3:2), and the subdominant (4:3). Each of these is assigned their major triads (frequency ratios 6:5:4). We have therefore 3 x 3 notes in total. But two notes are repeated twice; hence in fact 3 x 3 – 2 notes in total. Now, 2 of the obtained intervals turn out to be "semitones" $\frac{16}{15}$, and five of the obtained intervals turn out to be "full tones", either $\frac{10}{9}$, or $\frac{9}{8}$. The ratio of the last two is, of course, the *syntonic comma*:

$$\frac{81}{80} = 1.0125.$$

The difference between the Pythagorean comma and the syntonic comma is called the *schisma*, and it is equal to about 1.93 cents. See Benson (2006: Section 5.8) for more information on *diaschisma*, the *great diesis*, the *septimal comma*. Historically, an important fact is that:

$$\left(\frac{81}{80}\right)^{\frac{12}{11}}$$

is almost equal to the Pythagorean comma. This was employed by Johann Philipp Kirnberger who gave a recipe to tune an instrument in equal temperament.[8] Thus, we need to divide each of the intervals into two parts, except the two smaller intervals remain intact. This gives totally 12 notes in an octave, because 2 x (3 x 3 – 2) – 2 = 12. There is no need to plumb deeper into this subject of just intonation; the variety of these is huge (Benson 2006: Section 5.5). These two extractions of a number "12" are completely independent: the second construction incorporates a fifth overtone into the theory, while the first one does not.

The final *a posteriori* opportune fact is that this number 12 (whose arithmetic properties are not considered or required in advance in the previous two examples) turns out to be a composite number, and even its Euler function value is small: $\varphi(12) = 4$. In other words, out of all natural numbers smaller than 12, only the numbers 1, 5, 7, 11 are relatively prime to 12. This gives a circle of semitones (1), a circle of fourths (5), a circle of fifths (7), and a circle of major sevenths (11). The fact that one of these circles is related to a number 7, which appears in the fraction $\frac{7}{12}$, follows from basic results about continued fractions, meaning that this particular circle coincides with the circle in which tonalities are arranged in order of harmonic proximity.

Starting from this point, cultural, aesthetic, technical, social, artistic factors enter the scene, and questions of a very different a nature (albeit, still highly mathematical) emerge – how to distribute these 12 notes in an octave. A classes of just, meantone, irregular temperaments each split into a plethora of examples, while classes of Pythagorean and equal temperament consist of a single tuning each. These questions are dealt with in great detail in Benson (2006: Chapter 5). In the majority of cases, the precise tuning is a matter of theory rather than an artistic appeal. But in few cases – and this point needs a particularly convincing emphasis, since this highly motivates the study of temperaments solely due to artistic reasons – a scrupulous pre-tuning of an instrument gives static chords which carry meaningful emotional information. For example, American composer Michael Harrison released two CDs *From Ancient Worlds* (1992) and *Revelation* (2007). Music is played on a 24 tone 7-limt just scale (denominators of rational numbers appearing in tuning have only prime numbers 2, 3, 5, and 7 as factors). The booklet of the first CD contains the following explanation made by the author:

> Beginning in 1986, I spent two years extensively modifying a seven-foot Schimmel grand piano to create the Harmonic Piano. It is the first piano tuned in Just Intonation with the flexibility to modulate to multiple key centers at the press of a pedal. With its unique pedal mechanism, the Harmonic Piano can differentiate between notes usually shared by the same piano key (for example, C-sharp and D-flat). As a result, the Harmonic Piano is capable of playing 24 notes per octave. In contrast to the three unison strings per note of the standard piano, the Harmonic Piano uses only single strings, giving it a "harp-like" timbre. Special muting systems are employed to dampen unwanted resonances and to enhance the instrument's clarity of sound.[9]

All other tunings (Bohlen-Pierce scales, scales of Wendy Carlos, Harry Partch scales, and so on), can hit close to one of these three requirements (most commonly, achieve a good fifth), but not two of them, not to mention all three at once:

1) a good fifth;
2) accommodation of a fifth harmonic into a theory;
3) numerology – a number of notes in a scale being highly composite, or at least with several divisors.

And so, experimental tunings remain interesting musical adventures, attractive and important for mathematical investigations, but limited in their aesthetic or artistic appeal.

3. Mathematics of a single sound

Harmonic motion of a spring is a consequence of Hooke's law (a mathematical fact which states that a force, subject to the condition that a displacement is small, is a smooth function), and Newton's second law (force equals mass times an acceleration). Combining these both, one is led to a problem of finding a function which satisfies:

$$\frac{d^2f}{dx^2} = -f.$$

The latter differential equation is satisfied both by functions sin x and cos x. Whence the importance of trigonometric functions in acoustics comes from. This also shows that the main building blocks of a sound wave are sinusoidal sounds.

Fourier analysis in general is a vast generalization of Euclidean geometry and the coordinate method. Many basic results in Fourier analysis arise from analogous results in a 2-dimensional Euclidean geometry. For example, Parseval's identity in its simplest form is just a Pythagorean theorem: for two sides a and b and a hypotenuse c of the right triangle, one has $a^2 + b^2 = c^2$. Fourier analysis deals with many different mathematical spaces and functions, but as far as sound is concerned, it tells that a sufficiently "nice" periodic function $f(x)$ (say, a function with a smooth derivative, and a period 1) can be expressed as a converging trigonometric series:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\pi x) + b_n \sin(n\pi x)).$$

The level of smoothness is directly related to the order of vanishing of the coefficients. In musical terms, a discontinuous signal (like a sawtooth wave) has a very rich spectrum, and the weight of each component falls-off very slowly. Such signals are useful in a subtractive synthesis. A very smooth sound, on the other hand, has only a few strong initial (or further) harmonics. This is a relative observation, since a rate of decay of amplitudes of harmonics minds only an asymptotic behavior, while in practice, due to frequency range hearing limitations (20-20.000 Hz for young adults), only a finite amount of overtones do matter.

Two important topics which are adjacent to the one just described, and which are crucial in understanding sound, are damped harmonic motion (there is an additional frictional force proportional to a velocity; see Benson 2006: Section 1.10), and resonance (Benson 2006: Section 1.11). The first problem amounts to solving homogeneous second order ordinary differential equation with constant terms. The second problem asks to find and analyze a solution of a differential equation:

$$m\frac{d^2y}{dx^2} + \mu\frac{dy}{dx} + ky = f(x).$$

Here m is a mass, μ is a friction coefficient, k is a coefficient in Hooke's law, and $f(x)$ is an external excitation. The most interesting case arises when a spring is given a periodic external stimulus $f(x) = \sin(\alpha x)$, for some real number α. A group of soldiers marching on top of a bridge is an example. A thorough analysis can be carried out in this case, leading to such crucial notions as resonant frequency and (critical) bandwidth. Of course, a resonance on basilar membrane in a cochlea is a far more complicated phenomenon, but nevertheless tools from differential equations allow us to deal with it in a rigorous way.

4. Instruments

The basic application of mathematics in the study of musical instruments is the derivation of wave equations. The simplest case is a spring, giving an ordinary differential equation (ODE). However, this does not give rise to music. Wave equations are partial differential equations (PDE). For a vibrating string, it was first derived and solved by d'Alembert in his study *Recherches sur la courbe que forme une corde tendue mise en vibration* (Berlin, 1747). The same equation applies equally to chordophones and aerophones. Differences for various instruments arise in stating *initial conditions*. In case of chordophones, one set of initial conditions is given by the fact that a string is attached at both ends (whence comes the harmonic vibration), and the second set of initial conditions depends on the way the string is played: struck, plucked, or bowed. In case of aerophones, the wave equation describes the displacement of a single cross-section of an air tube, and also *the acoustic pressure* (the difference of the factual pressure with an ambient atmospheric pressure). The initial conditions depend on the pipe shape and whether the tube is open or closed at one of the two ends (Benson 2006: Section 3.5). This gives different initial conditions for a flute (a cylinder, both ends are open), a clarinet (a cylinder, only one end is open), and an oboe (a cone, one end is open).

For a membranophone, the wave equation is 2-dimensional, and it reads like:

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right).$$

Here $z = z(x, y, t)$ is a vertical displacement of a membrane, x, y are two coordinates along the membrane, t is a time variable, and the constant c depends on physical characteristics of a membrane (tension and mass density). The initial conditions depend on the form of the membrane. The spectrum of such a vibration is no longer harmonic, but is given by eigenvalues of a Laplace operator. This set of frequencies is subject to a few conditions concerning asymptotic behavior, but quite deprived of a structure otherwise. What music instrument manufacturers factually do, is that they control the first few eigenvalues in order to get a feeling of a pitch. For example, musical qualities of timpani, and how it produces a nearly harmonic spectra, is an extensive research topic in itself.[10] A curious question asked in this context was posed by Mark Kac, and it reads as "can one hear the shape of a drum"?[11] In other words, does the Laplace spectrum uniquely determines the membrane shape? The answer is "no", and it is given by the following example:[12]

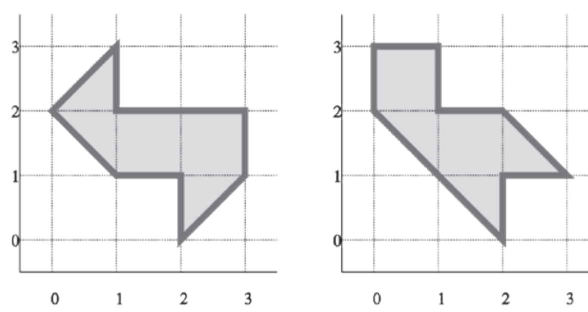


Figure 1. Drumheads which sound the same (mathematically: isospectral bounded domains)

Such drumheads will sound identically. Of course, this is a deep mathematical phenomenon in a plethora of analogous phenomena. Yet, as far as musicology is concerned, this oddity has a limited applicability, not to mention music itself, where this is of no noticeable importance. Nevertheless, it seems to me that the very knowledge of this field of research gives to musicologists a deeper understanding and greater appreciation of the very nature of musical instruments.

The equations for idiophones are much more complicated. For a xylophone, tubular bells and mbira (a lamellophone from Zimbabwe) the equation is a fourth degree in two variables. For a gong it is fourth degree in three variables. One can give equations for bells, singing bowls, and so on. A crucial attribute of a sound can be achieved by a process of formation of a shape of a wooden or metal bar, by the means of cutting an arc. For example, a set of harmonic frequencies 1:4:10 can be achieved in case of a marimba, and a set of frequencies 1:3:6 in case of a xylophone. Tuning of idiophones, consequently, is an interplay among mathematics and mathematical physics, material science, mathematical theory of a temperament, and instrument producing craftsmanship facilitated by an ages-long experience.

The topic which is allied with a theory of wave equations, is the formation of Chladni patterns on the surface of membranes or metal plates (Ashton 2003: 46). For example, a stadium plate, 70 cm across, 3215 Hz:[13]

Figure 2. Chladni pattern on a metal plate

This, too, has substantial musicological applicability. Moreover, it gives a graphic interpretation and an analytic decomposition of a sound, which musically is without any distinguishable pitch. It is important to emphasize that the topic of Chladni patters is pure mathematics, and it has an important application in the manufacturing of musical instruments.

An area of research, highly related to the investigation of aerophones, is the science of acoustics. This is applied to a construction of music halls, studios, and practice rooms. This science, too, has a firm foundation in mathematical physics. Like metal plates, closed spaces have their own set of resonant frequencies.

Finally, we wish to emphasize that this section describes only the latter, resonant frequencies of instruments. As mentioned, the quality and timbre of a sound as perceived by a listener, much more substantially depends on the initial transient part. Without it, the sound of a piano does not differ a lot from the sound of a flute. The transient part needs a very sophisticated form of mathematics to be described. This part of instrument science is far too technical for musicologists (Fletcher, Rossing 2008). It is only important to know about these complications. Practically one relies on specific experiences of professional instrumentalists and manufacturers.

5. Digital music, synthesis

The process of digitizing sound involves two unrelated procedures. The first is called bit depth, or resolution, prescribed for each sample. Namely, all possible amplitudes are encoded in a discrete set of values determined by bit information (for example, "16 bits" mean $2^{16} = 65536$ possible values). Certain information is necessarily lost in this process – for example, a very low-level sound is reported as a null sound. The second process is called sampling, which amounts to dividing one second into M equal parts, the so called *spc* (samples per second), and taking an amplitude value at each of the measured instances. As an aside, though many curves in an Euclidean plane can pass through three given points, if one limits the geometry of curves allowed, the set of all possibilities reduces. For instance, only one circle passes through three non-collinear points. Returning to signal processing, if one reduces the geometry of sound signals allowed, all sample values can be sufficient to reconstruct the initial signal uniquely. In particular, one has the following theorem of an utmost importance:

> **Nyquist–Shannon sampling theorem**. If a function $x(t)$ contains no frequencies higher than M hertz, it is completely determined by giving its ordinates at a series of points spaced $1/(2M)$ seconds apart. (Benson 2006: Section 7)

Of course, the applicability of this theorem is not limited to music; it is crucial in telecommunications, radars, sonars, seismology, image processing, and many other fields. The proof of this theorem relies on basic applications of a Fourier transform. A signal has its Fourier transform, and the latter determines (a sufficiently "nice") signal uniquely via an inverse Fourier transform. Now, for a sampled signal, the Fourier transform is periodic with a period $(\Delta t)^{-1} = M$. If the initial signal satisfies the Nyquist condition, this periodic function can be truncated, no overlapping over a duration of one period occurs, and the inverse Fourier transform gives the initial signal. Carrying out this in practice is a different matter, and it requires analogue filters of a high fidelity.

Dithering, lossless encoding (WAV files), encoding with compression (MP3 encoding requires mathematics, acoustics, psychoacoustics, and much more), MIDI, z-transform, digital filters, discrete Fourier transform, fast Fourier transform – these are just a few topics related to digital sound processing.

Sound synthesis is another huge topic, whose success depends on mathematics, musical background, and extensive experimentation. The simplest frequency modulation (FM) synthesis with two oscillators requires the knowledge of Fourier series of a function $\sin(\varphi + z\theta)$, given by:

$$\sin(\varphi + z\theta) = \sum_{n=-\infty}^{\infty} J_n(z)\sin(\varphi + n\theta).$$

Here $J_n(z)$ are the so called Bessel functions: they are crucial not only in FM synthesis, but also in the study of membranophones with a circular drumhead. In fact, two oscillators can produce only a limited variety of sounds. Yamaha DX7 synthesizers employ 6 oscillators to produce a sound, including various parallel and feedback (self-modulation) connections (Benson 2006: Chapter 8). One of the configurations ("algorithms") of six oscillators is shown in the figure 3 (an additive synthesis of a single oscillator, a frequency modulated oscillator, and an oscillator whose frequency is modulated by another oscillator with an additional feedback).

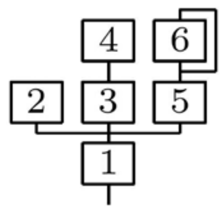


Figure 3. One of the arrangements of oscillators (an "algorithm") in Yamaha DX7 synthesizer

Moreover, each of these oscillators comes with their own envelope. Such configurations are too complicated to be described mathematically, though it can be done (Bessel functions yet again are crucial). Therefore, all meaningful sounds and soundscapes are a matter of practice and experimentation rather than a thorough mathematical pre-calculation. Concerning other synthesis algorithms (but not going into detail), one can mention granular synthesis, Karplus-Strong algorithm to generate plucked strings and drums[14] (Benson 2006: Section 8.5), phase vocoder, pitch envelopes, and so on.

6. Other appearances of mathematics in music

In this section, we briefly list a few other regions in the intersection of music and mathematics. This list is far from being exhaustive.

6.1. Euclidean geometry

In 1747, a Swedish organ maker Daniel Stråhle (1700–1746) proposed a geometric method to produce frets on a fingerboard of a fretted string instrument, in order to give a practical method to approximate equal temperament (*Music and Mathematics* 2003: Section 4). This method was dismissed by Jacob Faggot, a secretary of the Royal Swedish

Academy of Sciences. However, as was discovered by James Murray Barbour in 1957, Faggot greatly miscalculated the accuracy of Stråhle method.[15] The latter gives in fact a deviation of at most 3 cents in equal temperament. The essence of Stråhle approach is the mathematical formula:

$$2^x \approx \frac{24 + 10x}{24 - 7x}, x \in [0,1].$$

This approximation involves both approximation of functions, as well as approximation of real numbers (Diophantine approximation).

The construction itself is very elegant: the segment *QR* is divided into 12 equal parts of unit length, while *OQ* and *OR* have lengths 24, and *QP* has length 7 (see the figure 4 below). The point *P* is the middle of a string *MR*, and the positions of intersections with lines going from *O* to equally spaced points on *QR* give the position of frets on a string *MR*.

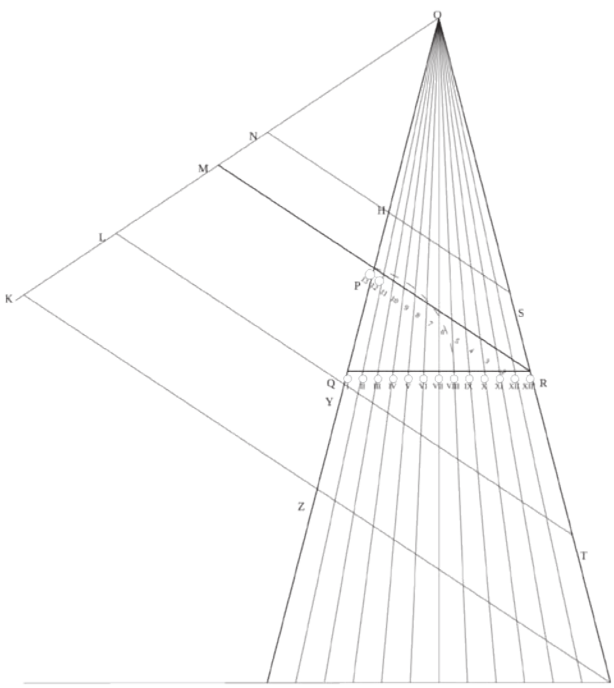


Figure 4. An illustration of Stråhle's method

6.2. Euclidean algorithm

In essence, this arises from the division formula with the remainder: for every pair of natural numbers *a, b* $\in \mathbb{N}$, there exists $c \in \mathbb{N}_0$ (natural numbers with 0), and r, $0 \leq r \leq b - 1$, such that $a = b{\cdot}c + r$. This guarantees that the unique factorization into prime numbers holds (the *fundamental theorem of arithmetic*). In musical terms, in a very rudimentary form, the latter tells us that there does not exist a tuning with all perfect fifths, or a tuning with all perfect major thirds, and so on. Compromises must be made. Also, the division with the reminder is the main tool allowing continued fraction expansion of rational numbers.

Recall that the continued fraction algorithm is the main apparatus which allows a build-up of a tuning system, which gives close approximation to a set of pre-selected intervals (most commonly, fifths).

6.3. Combination tones

When two loud sounds of frequencies *f* and *g* are played simultaneously, one can perceive a sound corresponding to the frequency $f - g$. This phenomenon was discovered by German organist Georg Andreas Sorge (1744), French physicist-lawyer Jean-Baptiste Romieu (1753), and a famous Italian violinist and composer Giuseppe Tartini (who claimed to have discovered these sounds as early as 1714). Hermann von Helmholtz (1856) revealed that the sound corresponding to the frequency $f + g$ also can be heard, but much less distinctively. As it turned out later, this happens due to the masking effect. The latter phenomenon occurs because of asymmetry in the excitation of the basilar membrane, where pure tones can mask tones of higher frequencies more effectively than tones of lower frequencies (Auditory demonstrations 1987: Demonstration 9). These extra sounds are called combination tomes. The ones corresponding to the difference of frequencies are sometimes called Tartini's tones.

Differently from the phenomenon of beats which arise from a summation formula for trigonometric functions, combination tones are a more complicated matter, related to the so called quadratic non-linearity nascent in the auditory system. Combination tones are proved to exist employing tools from the theory of differential equations. Additionally, cubic non-linearities produce sounds with frequencies $2f - g$ and $2g - f$. Quadratic non-linearities involve an asymmetry in the vibrating system, while cubic do not. This implies that cubic combination tones can be perceived even at low volumes, while high volumes are needed to experience sum and difference tones. This is what does occur in practice.

6.4. Archicembalo

Archicembalo is a harpsichord-type musical instrument, described by Vicentino in 1555 (Benson 2006: Section 6.5).[16] Vicentino also coined the term itself. This instrument had two manuals. The second one was not for timbral differences, but rather to provide extra pitches. One of the tunings of archicembalo is almost identical to equal 31-part tuning (31-tet). The main interest of this tuning arises from the fact that its 18th note gives a very good approximation to the meantone fifth:

$$2^{\frac{18}{31}} \approx \sqrt[4]{5},$$

the difference being only 0.19 cents. Another tuning of archicembalo was proposed by Vicentino, giving some pure fifths and a limited number of triads in just intonation (6:5:4).

Figure 5. Reproduction of the archicembalo

6.5. Golomb ruler and Costa arrays

These are important in designing effective sonars.[17] Costa arrays and Golomb ruler were used by Scott Rickard to compose a piece of piano music called The Perfect Ping. It is a monophonic music amounting to 88 notes, one for each key of the piano. The notes are played in such a succession that the permutation of notes obtained (the initial set is arranged in the standard way, the lowest key A_0 being the first) gives rise to a Costas permutation. That is, in this particular case, numbers $(1, 2, \ldots, 88)$ are permuted, say, as $(i_1, i_2, \ldots, i_{88})$, so that numbers $|i_1 - i_2|, |i_2 - i_3|, \ldots, |i_{87} - i_{88}|$ are all distinct; numbers $|i_1 - i_3|, |i_2 - i_4|, \ldots, |i_{86} - i_{88}|$ are all distinct; and so on. Moreover, a rhythmical pattern is arranged according to a Golob ruler. The latter depends on an integer n (in our case, $n = 88$), and consists of a set of marks at integer positions along an imaginary ruler so that no two pairs of marks are the same distance apart. For example, one unit of a mark can be made equal to a sixteenth note. This is a rather exotic application of mathematics to music, and yet with a certain substance. Comparisons to Tom Johnson's composition *Music for 88* are enlightening (Povilionienė 2016: 131).

6.6. Change-ringing (*Music and Mathematics* 2003: Section 7)

The ringing of a bell is a model of a pendulum. If an amplitude is small, then one uses the approximation $\sin x \approx x$, and the process is harmonic. In reality though, bells need to produce intense sounds. Thus, amplitudes ought to be large, the simple model $\sin x \approx x$ does not apply anymore, and the process turns out to be rather slow: if a mathematical pendulum swings in such that way that it barely reaches a vertical position at its extreme, the time it takes to get there is infinite (for a simplified harmonic motion, it is finite). In fact, the true mathematical model of a pendulum depends on the so-called elliptic integrals.[18]

There is an essential difference in approach how the bells are rung in continental Europe (mainly Germany) and The United Kingdom. In Germany, there are up to 5 bells in a belfry. Their pitches are such that, even if hit simultaneously, they produce a harmonious sound. Each of the bells, consequently, can be swung independently, its period depends on bells' weight, and the whole collection produces music of diverse, complicated melodic and harmonic pattern. In the English tradition, a different approach has been historically followed, which allows even a larger number of bells to be used. At the moment, there are over 5400 sets of bells in England and British isles. These sets have from 5 to 12 bells. Additionally, there are two towers with 14 and 16 bells, respectively. All bells in a set have small weight differences. The task is to achieve a full circle ringing. Due to a huge mass and slowness, each bell can only be slightly accelerated or slowed down at one swing, so that strikes of two adjacent bells can be interchanged in time, but not strikes of more remote bells.[19] For example, changes in four bell ringing might look like this (the real scheme has many repetitions; the fourth bell always sounds last):

1234 → 2134 → 2314 → 3214 → 3124 → 1324 → 1234.

Now it is obvious that in order to produce all possible permutations, one needs to employ knowledge of the group theory. In particular: symmetric groups, cosets, graphs, combinatorial group theory. Musical theory is needed as well. The problem of change-ringing was solved by English bell-ringers more than two centuries ago. But roughly a hundred years ago mathematicians began developing the concepts and terminology that facilitated elucidating all the theory in a group-theoretic language.

6.7. More examples

We finish this section by briefly listing several other topics in the intersection of music and mathematics. Now it should be clear that it is virtually impossible to provide an exhaustive list. The variety of mathematical tools employed in the study of these interactions amounts to only a small fraction of machinery developed in mathematics. Nonetheless, this variety is highly impressive and manifold.

- *Algorithmic or computer assisted composition.* To compose music, one writes an algorithm and uses numerical (or symbolic) data of various origins. Data can arise from dynamical systems, Markov chains, chaos, Turing machines, graphics, and so on. One of the most convenient tools to implement such composing is the programming language Csound (The Csound book 2000). An example of graphical data is a fractal (*Music and Mathematics* 2003: 163). The main aesthetic and artistic problems of composing with fractals arise from the lack (or presence of only a rudimentary) long term memory and musical form. To overcome these issues is a difficult artistic and programming task. Section 2.1 in Part 3 of the monograph *Musica mathematica* (Povilionienė 2016) is a neat description of many scenarios for human-computer interactions, and balances of contribution, in the composition process. For example, one makes distinction between CAC (computer-assisted) and AC (algorithmic) composition. Also, section 2.3 presents some examples and a few more thorough analysis of musical pieces related to fractals, *a priori* or *a posteriori* (*Ziqquratu* (1998) by Šarūnas Nakas, *Fractals* (1999) by Vytautas J. Jurgutis, among others).

- *Microtonality* (Jedrzejewski 2004). Relation of microtonal music to projective geometry (*Music and Mathematics* 2003: 149).
- *Statistics* (Beran 2003). The application of statistical methods in musicology is a subject far too broad to be even superficially portrayed. The following are being used: algebraic methods, various probabilistic distributions and tools from probability theory, data mining, time series, Markov chains, cluster analysis.
- *Riemannian and neo-Riemannian theory* (Jedrzejewski 2006: Chapter 4). This collection of ideas in musical theory are bound by and characterized by relating harmonies (triads and dissonant chords) directly to one another, without reference to the tonic. Putting aside issues of melody and phraseology, also applications of the theory in the study of late Romantic music (Ferenc Liszt, Anton Bruckner, and others, also the works of the late Renaissance composer Carlo Gesualdo), the theory also needs tools from group theory: group actions, centralizers, dihedral groups, combinatorial group theory, modular arithmetic, commutants, and so on. Naturally, the groups having a musical meaning are of a very special kind: for example, the dihedral groups $\mathbb{D}_{12}$ and $\mathbb{D}_{24}$. Thus, the mathematical machinery needed is more of an exercise in nature. Yet, the main motivation comes from the richness and diversity of musical material.
- *Emotional dichotomy of an effect of a minor–major triads.* Emotional differences evoked by playing a just minor (frequency ratio 15:12:10) and a just major (6:5:4) triads are in part a consequence of mathematical calculation (the greatest common divisor, the smallest common multiple, reinforcement of overtones), psycho-acoustics (the missing fundamental principle), and complicated evolutionary mechanisms which developed over time in humanoids (and other mammals) in the process of reaction to and extraction of vital information from low-pitched and high-pitched sounds.

7. "Musica Mathematica"

Roger Penrose classifies physical theories (by diminishing level of maturity and relevance related to describing our physical world) as "SUPERB", "USEFUL", "TENTATIVE", and "MISLEADING".[20]

Yet, as far as an imagination and worldview are concerned, falsifications, mystifications, or pseudo-sciences are still highly relevant. The false Ptolemaic system (geocentric model) was replaced by the Heliocentrism. Seven known moving bodies of the Sky were postulated as being an eternal quantity, and yet this changed after Galilei Galileo discovered four moons of Jupiter (1610), and William Herschel discovered Uranus (1781). Mathematicians of Antiquity developed a remarkable mathematical theory of Statics, but had a vague understanding of Dynamics. One can name topics in medicine (epidemiology), Biblical scholarship (Kabbalah), astronomy (flat Earth), chemistry (philosopher's stone), where misleading, unjustified or directly false theories were replaced by "SUPERB" ones afterwards.

Yet, as mentioned in Section 1, this attitude is not always uncontestably relevant. Even in linguistics one observes typical reflections of this dichotomy: some synthetic languages turn into more analytic, acquiring new lexical and syntactic tools, but losing certain morphological and grammatical features on the way.

This is exactly the attitude which makes **Part 1** of the monograph, called **"A retrospective of the Tradition of *Musica Mathematica*"**, not obsolete, not merely a historic overview, but an inspiring and useful summary, even for contemporary composers and scholars, of the interaction between music and mathematics. This part covers periods of Antiquity, the Middle Ages, Renaissance, and Baroque. The author exhibits a very broad erudition, presenting examples found in the Classical Antiquity (Aristotle, Zeno of Elea, Euclid, Vitruvius, Plato, Nicomachus), Christian Middle Ages (St. Augustine, St. Thomas Aquinas), and Renaissance (a plethora of authors). This is done without any cultural or religious bias: the list of religions concerned includes Hellenistic, Roman religions, Judaism, Christianity, various gnostic traditions. The author shows how mathematical facts taken from astronomy ("orbits of planets are circular", "there are seven moving bodies in the Sky"), linguistics ("seven vowels in Greek alphabet"), cosmology and cosmogony ("four elements in various proportions make up Cosmos"), mythology (Greek deities, goddesses, and Pantheon), were all interpreted in musical terms: ancient Greek musical modes, sound frequency ratios, note durations, other aspects of vertical or horizontal musical structure. The advancement of material is not chronologic, but rather logical and structural. It is a pity that the mention of Islamic Golden Age (8th c.–14th c.), highly relevant to the topic, is only fragmentary. For example, Ibn Sīnā (c. 980–1037) is mentioned once on p. 17, and Mūsā al-Khwārizmī (c. 780–c. 850) is mentioned on p. 223. Also, there is no information regarding music - mathematics relations in East Asia. For instance, He Chengtian (370–447) described an approximation to equal 12-part temperament in 400 A.D.,[21] while Zhu Zaiyu (1536–1611) solved a problem of exact equal temperament in 1584. The last method itself was very likely brought back to the West afterwards. This is mathematics, but it is needless to say that the amount of work done in China in the spirit of Chapter 1 of the monograph is far more abundant than purely technological-mathematical-musical investigations. Yet surely, such an expansion of geographic-cultural coverage is beyond scope of any reasonably sized study. There are few mathematical topics touched upon in Part 1, including Fibonacci and Fibonacci-type sequences, the Golden Ratio, various means (arithmetic, geometric and harmonic), combinatorics.

The title of **Section 2** of **Part 1**, namely, **"Semantic Interpretation of the Interaction between Music and Mathematics: Mystic Ages and the Sacral Baroque"**, is almost self-explanatory. It deals with numerology, Kabbalah, magic squares and hexagons (a purely mathematical topic with exterior symbolic-mystic interpretations, which is the most interesting part of the subject), alphabetic numerology.

A magic square of dimension n is a $n \times n$ grid of unit squares, filled with numbers $\{1, 2, ..., n^2\}$ in such a way that all rows and columns sum up to the same number, necessarily equal to:

$$S(n) = \frac{n(n^2+1)}{2}.$$

For example, if $n = 4$, the magic square present in the celebrated engraving *Melencolia I* (1514) by Albrecht Dürer is as follows:

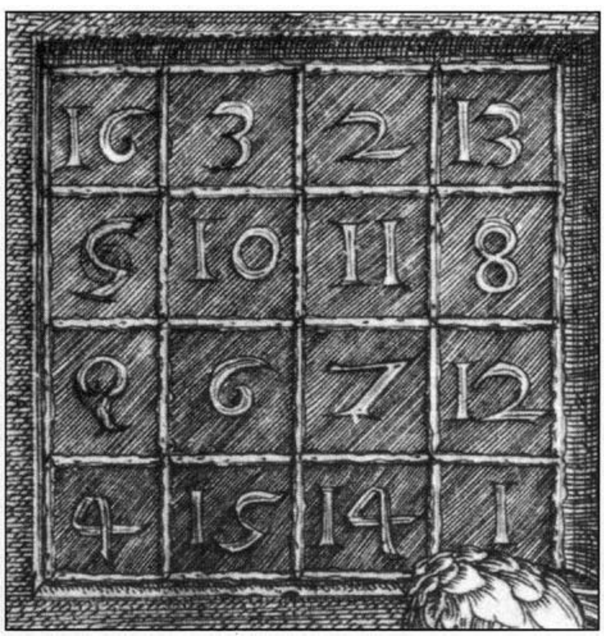


Figure 6. A close-up of the 4x4 magic square in Albrecht Dürer's *Melencolia I*

The engraving itself is analyzed on p. 172–174. This is done in relation to Dmitri Smirnov's cycle of pieces for piano *Two Magic Squares* (1971). The presented magic square has even more symmetries than required (mind sums of numbers in all 4 corner 2 x 2 squares of the large square, and so on), therefore gaining the name of "especially magical", "pan-magical" or "devilish" magic square. Smirnov translated this configuration of 16 numbers into a collection of 16 musical chords (p. 174). There are a few more peculiarities of the "devilish" square. The conjunction of numbers 15 and 14 seen on the bottom-row is the year "1514" when the work was made. Moreover, though the fact that $S(n) = 34$ in case $n = 4$ is a mathematical corollary, yet, if we attach to the decimal system, then $3 \times 4 = 12$ gives the number of notes in the tuning, and $3 + 4 = 7$ gives the number of notes in a diatonic scale. More complicated magical squares on p. 55 (where n ranges from 3 to 9) are being related to 7 moving bodies of the Sky, as taken from Agrippa (1533).

The second part of the monograph, titled "**The Renewal of Mathematical Techniques in Musical Compositions of the 20th and 21st Centuries**", though no new mathematical means are being introduced (the latter endeavor is the topic of Chapter 3), digresses from Part 1 significantly. Part 1 already has a coverage in mathematical-musical literature to a certain extent, while Part 2 mostly represents original research by the author. As we will see, many of the structural relations are implicit in titles or descriptions made by composers themselves, while some are hidden. In fact, going through the pages of Part 2, it is obvious that the balance of music and mathematics has shifted in the direction of music even more significantly.

The author divides the spread of traditions of *musica mathematica* of earlier epochs into two categories: formal-constructive, and semantic-symbolic. These two trends can go separately, or as a synthetic interaction. In cases where intentions of composers were encrypted, vague, or even deliberately withheld, it is very difficult to find the right key to "unlock" the piece. Part 2 presents many examples of such score analysis. It should be true that a high percentage of attempts to analyze musical scores failed to reveal features of this kind, whether the first or the second category were minded. Such compositions, naturally, were not included in the book. This further deepens the appreciation of the work executed by the author.

Examples of scores are presented where some musical parameters (rhythm, organization of pitches, bar structure, and so on) are tied to or are based upon standard sequences in mathematics: prime numbers, geometric progressions, and Mersenne numbers. The latter is the sequence of natural numbers $\{2^n - 1 : n \in \mathbb{N}\}$. One sometimes deals with a subsequence of the latter consisting solely of prime numbers. These should necessarily be of the form:

$$2^p - 1,$$

where p itself must be prime. Such numbers can still be composite. For example, $2^{11} - 1 = 2047 = 23 \cdot 89$. It is believed (this is still unknown) that there are infinitely many prime numbers among the Mersenne numbers.

A special treatment in the monograph is given to binary linear recurrence sequences, and especially to the most famous of them, the Fibonacci sequence (also to closely related Lucas sequence, and Serie Évangeliqué). Fibonacci numbers are defined from the seed values $F_1 = 1$, $F_2 = 1$, and the recurrence relation $F_{n+2} = F_{n+1} + F_n$. This sequence is fundamentally related to the Golden section, given by:

$$\varphi = \frac{\sqrt{5}+1}{2} = 1.618033\ldots$$

Sometimes one deals with the number $\varphi - 1 = \varphi^{-1} = 0.618\ldots$ instead. The relation is established with the help of the de Moivre formula:

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}.$$

This immediately shows that the ratio $\frac{F_{n+1}}{F_n}$ tends to φ, as n tends to infinity.

In music, the paradigm of Fibonacci-golden section can be dealt with in a discrete way, via the Fibonacci sequence itself (or transformations of this sequence), or in a continuous way, by approximating a ratio of specific musical parameters with the golden ratio. The topic "music and the golden ratio" is an old and prolific subject, yet it cannot be called artificial, at least in relation to the majority of examples presented in the book.

To understand where the golden ratio stands, we need a short explanation. First, it should be noted that out of the continuum of mathematical constants, the majority of them are uncomputable – no Turing machine can generate their digits algorithmically. Some are highly arithmetic yet quite rare (the so called "periods", and also special values of important functions, like Bessel functions discussed in this paper already). Yet some mathematical constants are ubiquitous, they are definitely at the very entrance of the mathematical Platonic reality: the natural logarithm basis $e = 2.7182\ldots$ (mentioned on p. 235 in relation to Conlon Nancarrow's *Cannon* $\frac{e}{\pi}$, also in the same page in formula describing probabilities for Poisson distribution), the Archimedes constant $\pi = 3.1415\ldots$ (mentioned on the same page, too, and also on p. 16), Euler-Mascheroni constant $\gamma = 0.5772\ldots$, Catalan's constant $G = 0.9159\ldots$, and few others. Out of algebraic numbers, the most ubiquitous are $\sqrt{2}$ (mentioned also on p. 235 in relation to another canon by Nancarrow), $\sqrt[3]{2}$, and the golden ratio φ. While dealing with tuning, we mentioned the fundamental role continued fractions play in the theory of temperament. It is a classical fact that the majority of asymptotic phenomena in this field are dominated by the constant φ: this is an irrational number whose Diophantine properties are extreme (the poorest approximation properties). This gives another, albeit quite a collateral relation between the golden section and music.

Among φ or Fibonacci-related pieces, the author analyses, accompanying this with a careful graphical presentation of scores, Derek Bourgeois' Symphony for Organ, Op. 48 (1975), Steve Reich's *Clapping Music* (1972), Karlheinz Stockhausen's *Klavierstück IX* (1961), and mentions extensive research done by other authors previously: analysis of Béla Bartók's *Music for strings, percussion and celesta* (1936), György Ligeti's *Volumina* (1961–1962), and others.

It should be noted, however, that artificial attempts to tighten up certain musical data to obtain the golden section relation were also noted, like in the quotation by Leon Harkleroad (Harkleroad 2006):

> Some of the most misguided attempts to link music and mathematics have involved Fibonacci numbers and the related golden ratio.

Part of the appearances of φ are truly unconscious, another part of them are just an accident. Needless to say, Rima Povilionienė understands these ramifications perfectly. Once more: this book is about the Art, not Science.

Many other mathematical tools for formal-constructive musical purposes are available – like symmetry, geometry, combinatorics, transformations – and these are discussed in the book. One also encounters some deeper mathematical objects, like formal grammars (L-systems). For example, L-systems appear in the analysis of Tom Johnson's *Mersenne numbers* (p. 136). L-systems are intricately related to finite automata, and hit close to the orbit of Turing's machines, algorithms, uncomputability, and complexity. This profound part of mathematics in musical composition is touched upon only slightly in the book. Yet the amount of examples is sufficient to inspire a thoughtful reader to proceed further with his/her own research.

Section 2 of **Part 2**, titled **"Semantic Aspects of Music Compositions"**, is no less exciting, and points back to Part 1. A new feature which was not explicit (or even non-existent) in earlier epochs, is the so-called "Personalized Semantics in Music". The year 1514 in Dürer's engraving, or many well-known alphabetic-numeric-musical manipulations of Johann Sebastian Bach (p. 65–67) can be taken as personal information, yet not as personal as number 22 for Alban Berg (p. 184), telephone numbers of relatives for Balys Dvarionas (p. 185), or Vietnam-war related dates for George Crumb (p. 186).

If the previous part of the monograph dealt with mathematical techniques which were re-actualized, yet not alien to epochs of the Middle Ages, Renaissance and Baroque, **Part 3** of the study, **"Innovations of Mathematical techniques in 20th and 21st Century Music"**, as is a tautological way to state it, deals with innovations, new mathematical tools in musical composition. Part 3 is the shortest of all three, only 49 pages long, but it is closest in spirit to Section 1–6 of the current review. The author does not, as before, divide the material into formal-constructive and semantic-symbolic parts. Truly, the number of ways to look at the problem is so manifold, the amount of examples is so abundant, that Part 3, hopefully, is just a seed for future research, a pursuit for professional musicologists and mathematicians to collaborate on equal footing. The distinctions now arise whether innovations in composition are outcomes of technological advancement (manipulations with sounds, algorithms, digital world), pure mathematical advancement (L-systems, group theory), or both. For example, fractals were discovered long ago (like Julia and Fatou sets, and even long before that, like the Thue-Morse sequence), but only computers allowed us to delve into the astounding complexity of these structures. The topics discussed in Part 3 also include: graph theory, curves similar to space-filling curves (the Hilbert curve), chaos theory, tilings (a topic also hitting close to the "dark side of mathematics", that is, uncomputability and Turing machines), stochastics, Markov chains. For example, Section 2.2 lists some processes (many of which are non-integrable, in other words, they have no inner structure apart from, possibly, few invariants) that have relevance to music.

A curious anachronism, presented in Part 3, is Bach's celebrated *canon cancrizans* from *Musikalisches Opfer*, BWV 1079. The design of this canon is analogous to the geometry of Möbius strip, defined by mathematician August Ferdinand Möbius in the 19th century:

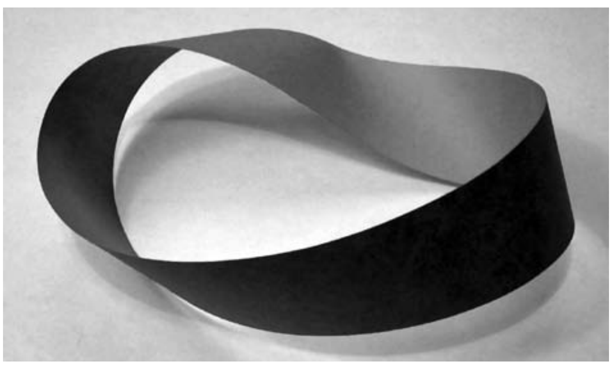


Figure 7. Möbius strip

Of course, this strip is not merely a graphical trifle. Among other applications in topology, it is an essential ingredient in classifying all 2-dimensional compact smooth manifolds. The emergence of the name "Bach" in this context is especially pleasing.

If we compare the Lithuanian original (Povilionienė 2013) with the English translation, then we can notice that the monograph was considerably revised, many parts of the narrative were expanded and clarified. Some diagrams, true, were left out. For example, graphic symbols for illustrations of George Crumb's cycle *Makrokosmos. Part 1* (Povilionienė 2013: 112), and illustration of Iannis Xenakis' *Nomos alpha* initial composing plan (Povilionienė 2013: 166), are missing in the English translation. All this may be due to copyright reasons. In the English translation, mathematical terminology is professional, only few typos have slipped into the text, due to miscommunication between mathematical editor (myself), and graphical editors; or just by an unfortunate accident. For example (p. 125), the use of "multiplication" (a process) instead of mathematically correct "product" (a result), or the phrase "scholastic music" instead of the correct "stochastic music" (p. 203, while on p. 204 the same expression is correct). There are few places with quite a vague mathematical phrasing, especially concerning topics related to asymptotic phenomena, like an infinite sequence, or a summation of an infinite series. Of course, this is not a drawback, since for musicologists without training in calculus standard mathematical routines may fail to be informative.

One tiny critique of the whole text concerns the interpretation of musical data and relating it to various mathematical objects in cases where a professional mathematician will fail to ascertain a relation. This concerns, however, only few examples. For instance, while analyzing Derek Bourgeois' *Passacaglia di Fibonacci* and its measure structure, the author explains it employing not only the Fibonacci sequence $\{F_n : n \in \mathbb{N}\}$ itself, but also the sequence $\{12 - F_n : n \in \mathbb{N}\}$, and also the set made of all $F_n + F_m$. This seems to be rather arbitrary and forced. As an indulgence to the author, the very title of the Bourgeois' piece (which is the third part of the Symphony for Organ) prescribes to look for interpretation in terms of the Fibonacci sequence.

The most unfortunate omissions, if one compares the Lithuanian text with the English translation, are two appendices. The first one is a short summary of mathematical facts and ideas. Though (as now must be clear from this Section and Section 1), this is not the most important aspect of the research by Povilionienė, hopefully there will be a second edition, which will include an even larger summary of necessary mathematical topics and notions. The second omission is a compendium of symbolic meanings of natural numbers, starting from 1, and going up to 1746 (internet search reveals that one of the interpretations of the latter number, in the tradition of Freemasonry, is that 1746 is the sum of the lunar number 666 and the solar number 1080). Curiously, the first omission in the sequence of natural numbers is $n = 32$. Interpretations vary from mathematical (prime numbers, perfect numbers), Christian (Biblical references of extremely diverse origin: number of strings in David's harp, number of Christ's wounds, etc.), geometric, psychological, linguistic (alphabets and counting, Greek, Hebrew, Latin, the tower of Babel), astronomical, alchemical, to simply historical, and so on. Hopefully, this Appendix will find its way into future works, especially the part of it which is related to music.

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2 See a book by Roger Penrose, *The Emperor's New Mind: Concerning Computers, Minds and The Laws of Physics* (Oxford University Press, 1989).

3 More about the clay tablets see the article "The Babylonian Musical Notation and the Hurrian Melodic Texts" by Martin L. West, published in *Music and Letters*, Vol. 75(2), May 1994, p. 161–79.

4 In literature:
Poetry changes with time, gets further away from its initial forms, and gets back again, but does not improve. Who would dare to say that the poetry of ancient Egypt, with its captivating comparisons, is worse than poems in the end of this collection (i.e., modern ones – G.A.)?
(Quotation taken from *Skiriama M. Pasaulio meilės lyrika* [*Devoted to L. World Lyrics about Love*], Kornelijus Platelis (compilation and afterward), Vilnius: Tyto Alba, 2015)

5 See, for example, an interview to Prof. Dr. Algirdas Burkštus on the alchemy discipline at the Vilnius University: https://www.tv3.lt/naujiena/269278/algirdas-brukstus-alchemija-zmogaus-vidaus-ir-isores-vienoves-atstatymas [last checked 2018 11 03].

6 Compare to mathematical linguistics:
However, the debt of mathematical linguistics to mathematics is chiefly the attitude and the way of thinking rather than any particular result. It is perhaps no accident that most graduate students in this field come from mathematics or logic rather than from linguistics.
(Quotation taken from Zellig Harris, "Mathematical linguistics", in: *The Mathematical Sciences. A Collection of Essays*, Cambridge, MA: MIT Press, 1969, p. 190–196)

7 See an article "Memory for Musical Attributes" by Daniel J. Levitin, in *Music, Cognition, and Computerized Sound* 1999: 2010.

8 Kirnberger wrote on equal temperament in the 2nd part 3rd division of his book *Die Kunst des reinen Satzes in der Musik* (Berlin, 1779).

9 I myself wrote reviews for the first and the second CD, emphasizing questions "what mathematics gives to aesthetics" directly: a review for CD *From Ancient Worlds For Harmonic Piano* see https://www.amazon.com/gp/customer-reviews/R2IF7HFJPTF30L/ref=cm_cr_dp_d_rvw_ttl?ie=UTF8&ASIN=B000QZRLHO; for CD *Revelation* see https://www.amazon.com/gp/customer-reviews/R2RH3VPPE14ICN/ref=cm_cr_dp_d_rvw_ttl?ie=UTF8&ASIN=B004FX8L1S [last checked 2018 11 01].

10 Davis, Robert E. *Mathematical Modeling of the Orchestral Timpani*, Ph.D. thesis, Physics Department, Purdue University, 1988.

11 More see an article "Can One Hear the Shape of a Drum?" by Mark Kac, published in *The American Mathematical Monthly* (Vol. 73(4), 1966, p. 1–23).

12 The figure published in the article "One cannot hear the shape of a drum" by Carolyn Gordon, David L. Webb, and Scott Wolpert (*Bulletin of the American Mathematical Society*, Vol. 27(1), 1992, p. 134–138).

13 More about Chladni patterns see the information presented by the Experimental Nonlinear Physics Group from the Department of Physics at the University of Toronto: https://www.physics.utoronto.ca/~nonlin/chladni.html [last checked 2018 11 03].

14 On Karplus-Strong algorithm see: Kevin Karplus, Alex Strong, Digital synthesis of plucked-string and drum timbres, *Computer Music Journal*, Vol. 7(2), 1983, p. 43–55.

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16 More about Vicentino's experiments see: Henry W. Kaufmann, More on the tuning of the Archicembalo, in: *Journal of the American Musicological Society*, Vol. 23(1), 1970, p. 84–94.

17 Sonars – acronym for SOund Navigation And Ranging. More see: Konstantinos Drakakis; Scott Rickard; James K. Beard; Rodrigo Caballero; Francesco Iorio; Gareth O'Brien; John Walsh. Results of the Enumeration of Costas Arrays of Order 27, *IEEE Transactions on Information Theory*, Vol. 54(10), 2008, p. 4684–4687.

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Santrauka

Nustatyti, kada matematika ir muzika, kaip žmogiškosios kūrybos sferos, pradėjo veikti vieną kitą, yra neįmanoma. Viena vertus, matematika galbūt vienintelė iš gamtos ir fizikinių mokslų kartais gali būti klasifikuojama kaip humanitarinė disciplina. Kita vertus, muzika – tai menas, arčiausiai ir esmingiausiai iš visų menų susijęs su matematika. Tad šie įvairialypiai ryšiai tikrai nestebina.

Pastaraisiais dešimtmečiais pasirodė daugybė knygų ir monografijų, kurių pavadinimuose figūruoja abu žodžiai „muzika" ir „matematika". Knygų autoriai – profesionalūs matematikai. Didelė dalis jų turi (tam tikro lygio) muzikinį išsilavinimą kaip teoretikai, kartais – kaip profesionalūs atlikėjai. Nežiūrint to, tenka pasakyti, kad, pirma, didžioji dalis autorių (ir tai yra natūralu) neturi itin gilių muzikos istorijos, literatūros, teorijos žinių ir nėra meistriškai įvaldę kompozicinių technikų. Antra, tekstuose dominuoja gamtamokslinis požiūris, kurį straipsnyje pavadinau *evoliuciniu*. Kitaip sakant, tai yra paradigma, jog tam tikra pažinimo sritis ilgainiui vis tobulėja, augdama ant ankstesnių žinių dirvos. Tai, be abejo, tinka gamtos ir fizikiniams mokslams, bet visiškai netinka kalbant apie menus. Kita problema – kokia yra šių monografijų tikslinė auditorija? Ar tos žinios naudingos kompozitoriams, muzikos teoretikams, aktyviems muzikos klausytojams? Kartais – tikrai taip. Bet jei kalbėsime apie didžiąją dalį teorinės medžiagos, tuomet tenka atsakyti – ne.

Rimos Povilionienės monografija „Musica Mathematica: Traditions and Innovations in Contemporary Music" (Peter Lang Academic Research, 2016, 288 p.) šiame kontekste yra svarbus darbas. Tai knyga, parašyta muzikologės, humanitarių mokslų daktarės, turinčios magistro lygio instrumentalisto (solinio fortepijono) išsilavinimą ir besilaikančios *kreacionistinio* požiūrio į meno reiškinius. Pastaruoju vardu straipsnyje vadinu paradigmą, jog sąmonės skaidrėjimas ir tamsėjimas, mokslinis pažinimas ir dvasinės atvertys, vertybių ir atskaitos taškų kitimas (ne progresas), tobulėjimas ir degeneracija, racija ir iracionalybė – viskas vyksta vienu metu. Savaime suprantama, šių skirtingų požiūrio taškų – evoliucinio ir kreacionistinio – problematiką gerokai supaprastinu. Pavyzdžiui, kiek reikia tekstologijos, kriminalistikos, chemijos, istorijos, kultūros, logikos ir kt. žinių, nustatant muzikos rankraščių autorystę, iš kitos pusės, galima prisiminti platų kultūrinį, filosofinį, religinį Renesanso laikų matematikų išsilavinimą. Vis dėlto konfliktas,

kylantis (skaitant matematikų darbus) iš *muzikos esmės* ir *dėstomos teorijos* prigimčių fundamentalių skirtumų, yra itin ryškus.

Tad šio straipsnio tikslai yra du. Pirma, orientuojantis į tuos muzikologus, kurie neturėjo universitetinio matematinio išsilavinimo, aprašyti daugybę pjūvių, kur matematikos ir muzikos kūrybos sritys kertasi – tai:

- derinimai ir temperacija (kokios priežastys lemia 12 dalių derinimo dominavimą, taip pat konsonanso–disonanso problematika);
- vieno muzikos garso matematika (harmoniniai svyravimai, Furjė analizė, slopinami svyravimai, rezonanso reiškinys);
- instrumentų matematinė fizika (bangos lygtys, pradinės sąlygos, Chladni raštai, pritaikymai instrumentų gamyboje);
- skaitmeninė muzika ir sintezė (*sampling'o* teorema, dažninė moduliacija, sintezės algoritmai);
- taip pat keli kiti pasirinkti pjūviai (styginių instrumentų su skirsniais gamyba, kombinaciniai tonai, Archicembalo, Costa masyvai ir Golombo liniuotė, varpų skambinimas Anglijoje ir t. t.).

Išsamus ir gilus visos šios naudojamos matematikos suvokimas nėra būtinas nei kompozitoriams, nei teoretikams. Vis dėlto (mano tvirtu įsitikinimu), bent paviršinis dalies reiškinių (kaip kad izospektrinių būgnų ar grandininių trupmenų) suvokimas nepaprastai naudingas tiek kuriant (netiesioginės inspiracijos), tiek analizuojant muziką.

Antrasis tikslas yra apžvelgti R. Povilionienės monografijos turinį ir parodyti, kaip šis turinys papildo ankstesniuose skyriuose minėtas muzikos ir matematikos sąsajas. Dalis knygos temų tikrai yra plačiai nagrinėjamos literatūroje. Bet svarbiausios monografijos dalys, ypač tos, liečiančios semantinius simbolinius *musica mathematica* aspektus, sąryšius su viduramžių, Renesanso ir baroko pasauliais, numerologija, taip pat įvairių laikotarpių partitūrų analizės, yra mažai nagrinėtos temos, ypač (pasikartosiu) mokslininko, kuriam pirmasis ir dominuojantis tikslas yra muzika. Trečioji R. Povilionienės monografijos dalis yra pati trumpiausia, tai lyg sąvadas, branduolys tolesniam moksliniam darbui, bet jau kartu profesionaliai jungiant tiek tiksliuosius mokslus, tiek XX–XXI amžių muzikos studijų sferas.